Age-period-cohort analysis in the 1870s

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Graphical representation

In a very short time span around 1870 a number of pathbreaking expositions of the use of graphical representations as a help in conceptualizing mortality measurement were published, mostly in German: Knapp, Zeuner, Becker, Lexis and Lewin. The dissertations by Brasche and Verweij also belong to this effort, and Perozzo delivered an impressive *finale*. The graphical representations were accompanied by mathematical theory, in particular Zeuner (1869) gave a representation analogous to the fundamental partial differential equation later associated with McKendrick and Von Foerster.

Zeuner (1869) studied the survival density $f(t,x)$ as function of time of birth t and age x defined by the property that

$$V(x) = \int_{t_1}^{t_2} f(t,x)dt$$

is the number of individuals born in $[t_1, t_2]$ who survived age x. Zeuner called the function $f(t,0)$ the *birth curve* and any function $x \to f(t,x)$ a *mortality curve.* The current time (or *census time*) $\tau = t + x$ is constant on lines in the (t,x) plane with slope -1. Zeuner's book contains a total of 27 graphs, most of them representing views of the manifold (we could call it the *Zeuner surface* or the *Zeuner sheet*) $z = f(t,x)$ in three dimensions.

Concrete graphical representation of individual lives were pioneered by Knapp, who simply plotted line segments from birth to death, stacked above a time axis. The later so obvious idea of combining Zeuner's and Knapp's ideas of the individual life line and a (cohort, age) diagram seems to have been achieved first by Becker, who defined the life line of a person born at time t and dead at age x, i.e. time $\tau = t + x$, as the horizontal line segment (t,t) to $(t, t+x)$. Thus, Becker had arrived at a $(t, t+x) = (t, \tau)$-diagram.

Lexis studied individual lives primarily in Zeuner's (t,x) plane, but also noticed that the three relevant time dimensions are not represented symmetrically there, and he therefore also proposed an *equilateral* diagram, where age, period and cohort are all on the same scale. The equilateral diagram was further developed by Lewin.

Today the "planimetric" representation of life lines, now called the Lexis diagram, is by far the most important of the graphical representations discussed above. However we use neither Becker's (time, cohort) = (τ, t)-diagram with horizontal life lines nor Lexis' (cohort, age) = (t,x)-diagram with vertical life lines, but rather the third possibility: the (time, age) = (τ, x)-diagram with life lines at slope 1. It is not easy to pinpoint the origin of this representation.

The Zeuner sheet and the McKendrick-Von Foerster equation

From the survival density $f(t,x)$ Zeuner derived the fundamental general result that the number of deaths in some region A of the (t,x) plane is given as

$$-\int_A\int \frac{\partial}{\partial x} f(t,x)dxdt \quad ,$$

which is equivalent to the differential equation

$$\frac{\partial}{\partial x} f(t,x) = -\varphi(t,x) \quad ,$$

where $\varphi(t,x)$ is the *mortality density.* Changing to the coordinates $(\tau,x) = (t+x,x)$ of the present-day "Lexis diagram", let $n(\tau,x) = f(\tau-x,x)$ and $\gamma(\tau,x) = \varphi(\tau-x,x) = n(\tau,x)\mu(\tau,x)$ with $\mu(\tau,x)$ the usual death intensity per individual alive at (τ,x). It follows that

$$\frac{\partial}{\partial \tau}\ n(\tau,x) + \frac{\partial}{\partial x}\ n(\tau,x) = -n(\tau,x)\mu(\tau,x) \quad .$$

This is the celebrated *McKendrick-Von Foerster equation*, see e.g. the survey by Keyfitz & Keyfitz (1997).

A technical report with details and references is available on the Internet (Keiding, 2000).

REFERENCES

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RÉSUMÉ

On décrire le développement rapide pendant la période 1868-80 des méthodes graphiques et mathématiques pour étudier la mortalité par l'âge, le temps et la cohorte.