Asymptotic density in coalescing and annihilating random walk

Harry Kesten
Department of Mathematics
Malott Hall, Cornell University
Ithaca NY 14853, USA
kesten@math.cornell.edu

This is mostly joint work with **J. van den Berg**.
Annihilating and coalescing random walks were studied as simple interacting particle systems by Bramson and Griffeath (1980), and by Arratia (1981). They considered the following systems. Particles move according to a continuous time random walk on $\mathbb{Z}^d$. The particles only interact when a particle at some site x jumps to a site y which already contains a particle. At this time, the two particles annihilate each other and disappear from the system, or they coalesce to only one particle at y, which continues with its random walk until it again coincides with another particle. The former system is called *annihilating random walk* and the latter system is called *coalescing random walk*. These systems first arose as duals to the "anti-voter model" and the "voter model" and were used as tools to analyze the voter model. Further motivation comes from models for chemical reactions. For chemical reactions one often considers particles of two types and allows only particles of different types to annihilate each other (or to form an inert compound). Here we shall restrict ourselves to systems with particles of one type only.

Usually one starts at time 0 with one particle at each site of $\mathbb{Z}^d$, although some results are valid for more general translation invariant initial states. It is further common to let the particles move according to continuous time simple random walk. That is, the particle jumps at the times of a rate 1 Poisson process, and when it jumps from position x, then it jumps to any one of the $2d$ neighbors of x with probability $1/(2d)$. For this version of the model, Bramson and Griffeath and Arratia found the asymptotic behavior of

$$p(t) := P\{\mathbf{0} \text{ is occupied at time } t\}.$$

For the coalescing random walk in dimension $d \geq 3$ one has (Bramson and Griffeath (1980))

$$p(t) \sim \frac{1}{\gamma_d t}, \tag{1}$$

where

$$\gamma_d = P\{\text{simple random walk in } \mathbb{Z}^d \text{ never returns to the origin after first leaving it}\}. \tag{2}$$

For annihilating random walk in $d \geq 3$ Arratia (1981) shows

$$p(t) \sim \frac{1}{2\gamma_d t}. \tag{3}$$

These articles also find the asymptotic behavior of $p(t)$ for $d = 1$ or 2, but we shall only be concerned with $d \geq 3$ here. It seems that the proof of Bramson and Griffeath and Arratia is very dependent on their specific assumptions and does not allow much, if any, variation in the interaction rules. On the other hand, there is an intuitively appealing, heuristic derivation of (1) and (3), which will be shown in our talk.

The main purpose of our paper is to turn those heuristic arguments into a rigorous and quite robust proof. To demonstrate the usefulness of our method we consider the following model: We start with one particle at each site of $\mathbb{Z}^d$. $\{S_t\}_{t\geq 0}$ is a continuous time random walk on $\mathbb{Z}^d$, with jump distribution $q(\cdot)$. We assume that

$$q(\mathbf{0}) = 0 \tag{4}$$

and that

$$\text{the support of } q(\cdot) \text{ contains } d \text{ linearly independent vectors.} \tag{5}$$

The motion of the particle which starts at x is distributed like $\{x+S_t\}_{t\geq 0}$. The particles interact only at times when a particle jumps to a site at which there are a number of other particles present. If there are j particles present, then the particle which just jumped coalesces with one of these j particles with probability p_j. We prove the following theorem for this model:

Theorem 1 *Assume that*

$$p_0 = 0, \quad p_1 > 0, \tag{6}$$

and that

$$p_j \textit{ is increasing in } j. \tag{7}$$

Assume further that the particles perform continuous time random walks which are distributed as translates of $\{S_t\}$, *that* (4) *and* (5) *are satisfied and that*

$$ES_t = t\sum_{y\in\mathbb{Z}^d} yq(y) = \mathbf{0} \textit{ and } \sum_{y\in\mathbb{Z}^d} \|y\|^2 q(y) < \infty. \tag{8}$$

Finally, assume $d \geq 6$. *Then in the above coalescing model there exists a* $\zeta = \zeta(d) > 0$ *such that*

$$p(t) - \frac{1}{C(d)t} = O\big(\frac{1}{t^{1+\zeta}}\big), \quad t \to \infty, \tag{9}$$

with

$$\begin{aligned} C(d) &= p_1 \sum_{m=0}^{\infty} (1-p_1)^m P\{S^\sigma_\cdot \textit{ returns exactly } m \textit{ times to } \mathbf{0} \\ &\qquad\qquad \textit{after first leaving it}\} \\ &= \frac{p_1\gamma}{1-(1-p_1)(1-\gamma)}, \end{aligned} \tag{10}$$

where $S^\sigma_\cdot$ is the difference of two independent copies of $S_\cdot$, and γ is the probability that $S^\sigma_\cdot$ never returns to the origin after first leaving it. Also

$$E\{\textit{number of particles at } \mathbf{0} \textit{ at time } t\} - \frac{1}{C(d)t} = O\big(\frac{1}{t^{1+\zeta}}\big) \tag{11}$$

and

$$P\{\textit{there are } \geq 2 \textit{ particles at } \mathbf{0} \textit{ at time } t\} = O\big(\frac{1}{t^2}\big), \quad t \to \infty. \tag{12}$$

In a forthcoming article with J. van den Berg we reduce the dimension requirement to $d \geq 3$ (which is best possible) under the additional condition

$$p_M = 1 \text{ for some } M < \infty. \tag{13}$$

Stephenson (1999) in his thesis considered similar models with (6) replaced by

$$p_i = 0 \text{ for } i \leq k-1, \; p_k > 0. \tag{14}$$

In this case one finds that the density $p(t)$ behaves asymptotically as

$$\frac{1}{C^*(d)t^{1/k}}, \tag{15}$$

for some explicit constant C^*.

In Kesten (2000) we have considered similar models but now coalescence or annihilation already takes place when a particle comes within distance one of another particle. This mimicks better what one would like to model when particles of a nonzero size perform Brownian motions in $\mathbb{R}^d$, instead of a random walk on $\mathbb{Z}^d$. This modification creates many technical difficulties but leads to results similar to (11).

REFERENCES

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RESUME

Nous étudions une système de particules où chaque individu exécute une marche aléatoire fixée sur $\mathbb{Z}^d$ en temps continu. Ces marches sont indépendantes sauf quand une particule saute à un site x qui est déjà occupé. S'il y ci trouve déjà j particules, alors la particule qui saute se combine à une des j particules à x avec probabilité p_j. Sous des conditions convenables nous donnons le comportement asymptotique de $p(t) := P\{\mathbf{0} \text{ est occupé à temps } t\}$.