Efficient Layout Techniques in Parameter Design Experiments

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Abstract

In some applications the inner-out array format for parameter design experiments will be too costly to conduct. Instead of using a single array for both control and noise factors, Taguchi proposed the compound noise technique to reduce the size of experiments. It works by selecting a few "extreme" conditions in the noise factors and "compounding" them into a single factor. We study its performance through theoretical analysis, real data example, and simulation.

Robust parameter design has become a commonly used technique for improving product and process quality, manufacturability, and reliability at low cost. In a parameter design experiment, factors are divided into two types: control factors and noise factors. Taguchi's (1986) main innovation was the proposal for reducing noise variations by exploiting control-by-noise interactions. To achieve it Taguchi advocated the use of the *cross array* that consists of a product of two orthogonal arrays. Therefore, the total experimental run size is the product of control array run size, *n*, and noise array run size, *m*. In practice, n*m can be prohibitively large. To reduce run size or cost, Taguchi proposed the *compound noise factor* technique, which replaces the noise array by a compound noise factor that has only a few levels, typically two, resulting in the savings of n*(m-2) runs. The following questions are raised naturally: Is the compound noise factor technique generally effective? If not, when can it be used? Under what conditions dose it work? In this paper, some theoretical and empirical work will be presented to address these issues.

1. Extreme Settings of A Compound Noise Factor

The reason for using the compound noise factor technique is to save experiment runs. The original idea is simple, that is, to use two extreme compound noise factor settings to capture the variances caused by the noise variations. Theses two extreme compound settings maximize and minimize the response. Therefore, extreme settings play a key role.

Theorem 1. Suppose that the model is given by

$$y = \mu + \sum_{i=1}^{l} \alpha_i x_i + \sum_{i<j} \alpha_{ij} x_i x_j + \sum_{j=1}^{m} \beta_j z_j + \sum_{i=1}^{l} \sum_{j=1}^{m} \gamma_{ij} x_i z_j \quad (1)$$

where z_j is symmetric about zero, If $\mathbf{z}^* = (z_1^+, z_2^+, \ldots, z_m^+)$ and $\mathbf{z}_* = (z_1^-, z_2^-, \ldots, z_m^-)$ are extreme settings, then they must be opposite settings, i. e., $\mathbf{z}^* = -\mathbf{z}_*$.

Theorem 2. Suppose that model (1) holds with nonzero β_j. If $\mathbf{z}^*$ and $\mathbf{z}_*$ are extreme settings, then sgn(z_j^+)=sgn(β_j) and $|z_j^+| = \max_{z_j \in R_j} |z_j|$, where R_j is the range of z_j, *j=1, 2, ..., m.*

Theorems 1 and 2 imply an elegant result that only opposite settings with sgn(z_j^+)=sgn(β_j) and $|z_j^+| = \max_{z_j \in R_j} |z_j|$ can be extreme settings.

Theorem 3. Suppose model (1) holds. Then extreme settings exist if and only if

$$|\beta_j| \geq \sum_{i=1}^{l} |\gamma_{ij}|, \ \textit{for } j=1, 2, \ldots, m, \quad (2)$$

where z_j is symmetric about zero and x_i is rescaled to be in *[-1,1]*.

2. A Theory for Compound Noise Factor

The previous part shown that for model (1) the effective use of a compound noise factor must involve the use of extreme settings. In practice, extreme settings may not exist. They may not work either when they do exist. The following parts discuss a set of conditions that guarantee the effectiveness of compound noise factor technique from the simple case of one control and m noise factors to the general case of l control and m noise factors. For simplicity, all noise factors are rescaled to be in $[-1, 1]$ with variance 1.

2.1 One Control Factor and *m* Noise Factors

Theorem 4. Under model

$$y = m + a x + \sum_{j=1}^{m} b_j z_j + \sum_{j=1}^{m} g_j x z_j \quad (3)$$

extreme settings identify robust settings (i.e., minimizing the true variance) if and only if

$$(\sum_{j=1}^{m} b_j g_j) * (\sum_{j=1}^{m} z_j^{+} g_j) > 0 \quad (4)$$

where z_j's are independent and symmetric about zero, and x has two levels at 1 and -1.

Theorems 3 and 4 gave a set of necessary and sufficient conditions under which the compound noise factor technique correctly identifies robust settings. Theorem 5 summarizes them in a single statement.

Theorem 5. Under model (2), the compound noise factor technique identifies robust settings if and only if

$$|b_j|^3 |g_j| \text{ and } (\sum_{j=1}^{m} b_j g_j) * (\sum_{j=1}^{m} g_j z_j^{+}) > 0 \quad (5)$$

where z_j's are independent and symmetric about zero, $z_j^{+} = \mathrm{sgn}(b_j) \max_{z_j \in R_j} |z_j|$, and x has two levels at 1 and -1.

2.2 *l* Control Factors and *m* Noise Factors

Theorem 6. Under model

$$y = m + \sum_{i=1}^{l} a_i x_i + \sum_{j=1}^{m} b_j z_j + \sum_{i \le j}^{l} a_{ij} x_i x_j + \sum_{i=1}^{l} \sum_{j=1}^{m} g_{ij} x_i z_j \quad (6)$$

and conditions

$$|\sum_{j=1}^{m} b_j \gamma_{ij}|^3 \sum_{k \ne i}^{l} |\sum_{j=1}^{m} g_{ij} g_{kj}|, \; i=1, 2, \ldots, l, \quad (7)$$

and

$$|\sum_{j=1}^{m} |b_j z_j^{+}| \sum_{j=1}^{m} g_{ij} z_j^{+}|^3 \sum_{k \ne i}^{l} |\sum_{j=1}^{m} g_{ij} z_j^{+} \sum_{j=1}^{m} g_{kj} z_j^{+}|, \; i=1, 2, \ldots, l, \quad (8)$$

extreme settings identify robust settings if and only if

$$(\sum_{j=1}^{m} b_j g_{ij}) * (\sum_{j=1}^{m} |b_j z_j^{+}| \sum_{j=1}^{m} g_{ij} z_j^{+}) > 0, \; i=1, 2, \ldots, l, \quad (9)$$

Note that the conditions (7) and (8) are sufficient but not necessary. A set of necessary and sufficient conditions are stated as follows.

Theorem 7. Under model (6), extreme settings minimize the variance if and only if equations

$$V_1 = \sum_{i=1}^{l} (\sum_{j=1}^{m} b_j g_{ij}) x_i + \sum_{i \pi k}^{l} (\sum_{j=1}^{m} g_{ij} g_{kj}) x_i x_k, \quad (10)$$

and

$$V_2 = \sum_{i=1}^{l} (\sum_{j=1}^{m} |b_j z_j^{+}| \sum_{j=1}^{m} g_{ij} z_j^{+}) x_i + \sum_{i \pi k}^{l} (\sum_{j=1}^{m} g_{ij} z_j^{+} \sum_{j=1}^{m} g_{kj} z_j^{+}) x_i x_k, \quad (11)$$

have the same minimizing solution.

3. A Simulation Study

In practice, the models in the theoretical study are only approximations to true models with random error. A simulation is studied to investigate the performance of the rules obtained in previous theorems when a random normal error is included in the model. Three models are chosen to represent three typical scenarios in which the conditions in Theorem 6 are satisfied, barely satisfied, or violated.

Model 1.

$$\hat{y} = 20.27 - 58.31x_2 - 29.96z_1 + 60.76z_2 - 76.89z_3 + 30.63x_1x_2 - 21.79x_1z_3 - 23.13x_2z_3 + e \quad (12)$$

The random error e is generated by drawing *2000* simulation samples from $N(0, \sigma^2)$. A variety of σ values are chosen to see how the percent of success (i.e., identifying robust settings) depends on σ. Take

$\sigma = (0.3|\bar{q}|, 0.5|\bar{q}|, |\bar{q}|, 1.5|\bar{q}|, 2.0|\bar{q}|, 2.5|\bar{q}|, 3.0|\bar{q}|, 3.5|\bar{q}|, 4.0|\bar{q}\}, 5.0|\bar{q}|, 6.0|\bar{q}|)$

where $\sum$ is the vector of the coefficients in the model.

The choice of σ in proportion to $\bar{q}$ is somewhat arbitrary. As σ increases, the model fit deteriorates as evidenced by the decrease in the R^2 values of Table 1. In this sense the choice of σ values represents a broad spectrum of situations. As expected, the percent of success (see third column of Table 1) is high for well-fitted models. As the model fit deteriorates (i.e., low R^2 values), the percent of success gets lower.

The second model is obtained by adding the term $-24x_2z_2$ to model (12) so that the resulting model (13) violates the conditions in Theorem 6.

Model 2.

$$\hat{y} = 20.27 - 58.31x_2 - 29.96z_1 + 60.76z_2 - 76.89z_3 + 30.63x_1x_2 - 21.79x_1z_3 - 24x_2z_2 - 23.13x_2z_3 + e, \quad (13)$$

The third and final model in (14) is obtained by adding two terms $-24x_1z_2$ and $-24x_2z_2$ to model (12) so that more conditions in Theorem 6 are violated.

Model 3.

$$\hat{y} = 20.27 - 58.31x_2 - 29.96z_1 + 60.76z_2 - 76.89z_3 + 30.63x_1x_2 - 24x_1z_2 - 21.79x_1z_3 - 24x_2z_2 - 23.13x_2z_3 + e, \quad (14)$$

The results in Table 1 shows that this model is worse than the second model. Unlike the two previous cases, the monotonic relationship between R^2 and the percent of success does not hold here. Because conditions in Theorem 6 are strongly violated, the conclusion of Theorem 6 cannot be applied here to predict the performance of the compound noise factor technique.

Table 1.

σ	R^2 in model 1	Success (%)	R^2 in model 2	Success (%)	R^2 in model 3	Success (%)
$0.3|\bar{q}|$	.99	100	.99	75.2	.99	40.3
$0.5|\bar{q}|$	.99	99.9	.99	75.8	.99	38.6
$1.0|\bar{q}|$	.97	94.3	.97	75.6	.97	47.4

$1.5\lvert\bar{q}\rvert$	.95	85.7	.95	68.4	.95	50.1
$2.0\lvert\bar{q}\rvert$	.92	78.3	.92	68.7	.92	49.8
$2.5\lvert\bar{q}\rvert$	.88	70.6	.88	63.1	.88	54.5
$3.9\lvert\bar{q}\rvert$	.84	67.4	.84	62.3	.84	53.3
$3.5\lvert\bar{q}\rvert$	.80	63.3	.79	58.7	.79	54.7
$4.0\lvert\bar{q}\rvert$	.76	58.0	.75	54.4	.75	50.2
$5.0\lvert\bar{q}\rvert$	.69	55.6	.70	54.5	.71	51.9
$6.0\lvert\bar{q}\rvert$	.64	54.9	.65	48.3	.66	46.8

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