Car Parking for Mathematicians

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In the classical random car-parking model of Rényi (1958), unit-length cars arrive sequentially at a roadside of of length L. Each car parks on the roadside at a position chosen uniformly at random subject to the obvious constraint of non-overlap with previously parked cars, until there are no available spaces left of length greater than 1 ('jamming' of the interval $[0, L]$). With the resulting (random) number of cars at jamming denoted $N_1(L)$, Rényi proved that $E[N_1(L)]/L$, the mean number of cars per unit length, converges to a limit α_1 as $L \to \infty$, and identified the limit α_1.

Physical interpretations are numerous. For example, the 'roadside' can represent an individual's diary, with 'cars' representing demands on that individual's time. Instead of an individual, one might have a mainframe computer, or a band of radio frequency, parts of which are allocated to successive customers. In these contexts, this setup is sometimes called 'online packing'. In another interpretation, the roadside represents a surface, onto which colloidal particles or proteins (the 'cars') arrive sequentially and adsorb. Interest from the physical sciences in such models is extensive; for example, a series of surveys appears in volume 165 of *Colloids and Surfaces A* (2000).

In this last interpretation (and others), it is clearly of interest to generalise to two dimensions. To do this, consider an $L \times L$ 'parking lot'. Suppose 'cars' (now perhaps better thought of as helicopters) are all unit squares (or some other fixed shape) and arrive sequentially at random positions in the parking lot. Each successive car is placed at a location distributed uniformly at random over the parking lot subject to the constraint of non-overlap with previously placed cars, until there is no available space left large enough to contain a new car (jamming). Let $N_2(L)$ denote the (random) number of cars parked at the jamming time. The mean number of cars per unit area is now $E[N_2(L)]/L^2$; again one expects it to converge to a limit α_2 as $L \to \infty$. Palásti (1960) provided a partial proof of this, but despite the numerous simulation studies, no complete proof appeared until Penrose (2001a). Palásti (1960) also conjectured that $\alpha_2 = \alpha_1^2$; this has not been proved or disproved, but simulation studies suggest that it is false.

This paper contains a similar result to that of Penrose (2001a) for an analogous *lattice adsorption model*, along with an associated central limit theorem. Both lattice and continuum models have been much studied in the physical sciences.

In the lattice adsorption model, particles arrive sequentially at random (uniform) sites in an $n \times n$ 2-dimensional lattice, denoted $B(n)$. Each adsorbed particle permanently occupies a single lattice site and excludes all adjacent lattice sites from being subsequently occupied. If a particle arrives at an excluded site, it is *rejected* and discarded; otherwise it is adsorbed. This continues until there are no available lattice sites left ('jamming' of the lattice). With the resulting (random) number of occupied sites at jamming denoted $N(B(n))$, we show that $N(B(n))$ satisfies a law of large numbers and and central limit theorem as $n \to \infty$. This is a special case of results in Penrose (2001b) for general lattice models; the proof here is simpler and provides extra information on error bounds. We note that a central limit theorem for *continuum* car parking remains unproven in more than 1 dimension.

Theorem 1 *There are positive constants $\alpha, \sigma_1, \sigma_2, \sigma_3, \ldots$ such that as $n \to \infty$, we have*

$$n^{-2}E[N(B(n))] \to \alpha; \qquad E\left[\left|n^{-2}N(B(n)) - \alpha\right|\right] \to 0, \tag{1}$$

and with Φ denoting the standard normal distribution function,

$$P\left[\sigma_n^{-1}\left(N(B(n)) - E[N(B(n))]\right) \leq t\right] \to \Phi(t) \tag{2}$$

*for all t. Also, $\sigma_n = \Theta(n)$ (in fact σ_n/n tends to a strictly positive constant), and for any $\varepsilon > 0$, there is an $O(n^{\varepsilon-(1/2)})$ error bound in (**??**), uniformly in t.*

Proof. The $n \times n$ lattice $B(n)$ will be called the *target set.* One could also consider an arbitrary target set B in Z^2. We may as well assume that one particle arrives at each site in B, and that the arrival times are given by a family of independent random variables $T(x), x \in B$, each of which is uniformly distributed on the unit interval $[0,1]$.

The variables $T(x)$ induce a directed graph $G(B)$ with vertex set B, as follows. For each pair x, y of lattice sites, draw a directed edge from x to y if and only if (i) x and y are adjacent in the lattice, and (ii) $T(x) < T(y)$. Once we have the directed graph $G(B)$, we can forget about the arrival times, and determine which particles are accepted by the following algorithm; a little thought shows that this algorithm faithfully reconstructs the set of lattice sites at which particles are accepted.

1. Accept all roots of the graph. Reject all offspring of roots.
2. Remove all roots and offspring of roots from the graph.
3. Return to step 1.

Now suppose we actually have independent uniform random variables $T(x)$ defined for all x in Z^2. These induce a randomly directed graph on the entire lattice Z^2, denoted $G(Z^2)$, by the same rule as before. Restricting this graph to any target set $B \subset Z^2$, we obtain graphs $G(B)$ defined on the same probability space for all target sets B at once; this is an example of *coupling*, a popular probabilists' trick. By following the above algorithm starting with the entire graph $G(Z^2)$, we also have a decision on which vertices to accept in the entire graph $G(Z^2)$. For each lattice site x in Z^2, let $I(x)$ take the value 1 if it is accepted in $G(Z^2)$, and 0 if not; let $I_B(x)$ take the value 1 if it is accepted in $G(B)$, and 0 if not. Define

$$N^*(B) = \sum_{x\in B} I(x); \qquad N(B) = \sum_{x\in B} I_B(x),$$

and observe that $N(B)$ is the number of adsorbed particles in a realization of the lattice adsorption model with target set B.

The variables $(I(x), x \in Z^2)$ form a stationary random field, and so have a common expectation, denoted α. Therefore

$$n^{-2}E[N^*(B(n))] \to \alpha; \quad E\left[\left|n^{-2}N^*(B(n)) - \alpha\right|\right] \to 0, \tag{3}$$

where the first limit is actually an equality, and the second limit follows from the Ergodic Theorem. To prove something similar for $N(B(n))$, we must estimate the difference $|N(B) - N^*(B)|$. Since this is bounded by $\sum_{x\in B} |I_B(x) - I(x)|$, we obtain

$$E[|N(B) - N^*(B)|] \leq \sum_{x\in B} P[I_B(x) \neq I(x)].$$

Suppose $x \in B$. The event $\{I_B(x) \neq I(x)\}$ cannot happen unless influence from outside B propagates along a directed path in the directed graph $G(Z^2)$, that starts outside B and ends at x. In other words, $\{I_B(x) \neq I(x)\}$ cannot happen unless the event

$$F_\gamma := \{T(x_0) > T(x_1) > \cdots > T(x_m)\}$$

occurs, for some sequence $\gamma = (x_0, x_1, x_2, \ldots, x_m)$, with $x_0 = x$, and $x_m \notin B$, and with each successive pair (x_i, x_{i+1}) adjacent. The number of sequences γ of this form is bounded by $4(3^{m-1})$, and the probability of event F_γ is $1/(m+1)!$. Therefore, if the graph distance from x to the complement of B is d, then

$$P[I_B(x) \neq I(x)] \leq 4(3^{d-1})/(d+1)! := Q(d), \tag{4}$$

which tends to 0 (rapidly) as $d \to \infty$. Hence, if $d(x, B^c)$ denotes the graph distance from x to the complement of B,

$$n^{-2}E[|N(B(n)) - N^*(B(n))|] \leq n^{-2} \sum_{x\in B(n)} Q(d(x, B(n)^c)),$$

which tends to zero as $n \to \infty$ (e.g. by the Dominated Convergence Theorem). Hence, (**??**) remains true with N^* replaced by N, completing the proof of (**??**).

Turning to the central limit theorem, choose a small constant δ with $0 < \delta < 1/2$. For $x \in B(n)$ let $S(x, n)$ be the square of side n^δ centred at x, and let $Y_n(x) := I_{B(n)\cap S(x,n)}(x)$.

Then $I_{B(n)}(x) = Y_n(x)$ unless influence propagates from outside $S(x,n)$ to x, i.e. unless there is a directed path of the graph $G(B(n))$ that starts outside $S(x,n)$ and ends at x. Hence $P[I_{B(n)}(x) \neq Y_n(x)]$ is bounded by the probability that there exists a sequence $\gamma = (x_0, \ldots, x_k)$ that satisfies $x_0 = x$, $x_i \in B(n)$ for all i, and $x_k \in B(n) \setminus S(x,n)$, such that event F_γ occurs. Such a path is of length $k \geq n^\delta$, and therefore as for the earlier estimate (**??**),

$$P[I_{B(n)}(x) \neq Y_n(x)] \leq Q([n^\delta]). \tag{5}$$

Set $N'(B(n)) := \sum_{x \in B(n)} Y_n(x)$. By (**??**), since $n^2 Q([n^\delta])$ tends to zero as n becomes large,

$$P[N(B(n)) \neq N'(B(n))] \to 0; \qquad E[N(B(n))] - E[N'(B(n))] \to 0. \tag{6}$$

By an argument that can be found at the end of Penrose (2001b), but which is not included here due to space constraints, there is a constant $c > 0$ such that for all large enough n,

$$\sigma_n^2 := \mathrm{Var}[N'(B(n))] \geq cn^2. \tag{7}$$

The advantage of using the variables $Y_n(x)$ is that they are only locally dependent: if A, B are subsets of $B(n)$ separated by a (sup-norm) distance of $d(A,B)$, the family of variables $(Y_n(x), x \in A)$ is independent of $(Y_n(x), x \in B)$ whenever $d(A,B) > 2n^\delta$.

It follows from this local dependence (see e.g. Baldi and Rinott (1989), Corollary 2) and (**??**) that $P\left[\sigma_n^{-1} \sum_{x \in B(n)} (Y_n(x) - EY_n(x)) \leq t\right] \to \Phi(t)$ for all t, and combined with (**??**) this gives us (**??**). This approach also yields the error bound claimed.

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RESUME

On laisse tomber des voitures, une à une, sur un parking ayant la forme d'un réseau de dimensions $n \times n$. Chaque voiture occupe un site du réseau, empêchant l'occupation ultérieure des sites voisins. Chaque voiture est placée sur un site du réseau choisi au hasard et uniformément de ces disponibles, jusqa'à ce que le parking soit bloqué. Nous prouvons une loi spatiale de grands nombres et de théorème limite centrale pour le nombre final de voitures garées quand la taille de réseau devient grande.