Modelling Stochastic Volatility via Autoregression

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Let us consider the following model

$$y_t = m + s(1 + d\,u_t)\,z_t, \qquad (1)$$
$$u_t = g\,u_{t-1} + h w_t, \qquad (2)$$

where y_t is the return of the asset, σ, δ, g and h are unknown parameters associated with the return process, and w_t and z_t are independent standard Gaussian white noises. The background for consideration of stochastic volatility u_t, modelled by AR(1), is inspired by its continuous time counterpart, which is an Ornstein-Ublenbeck process, $dY = mt + s\,(1 + dU)dZ$, where

$$dU = -gUdt + h\,dW.$$

The similar continuous time models were first considered by Stein and Stein (1991) and Heston (1993) showed that the model addresses the derivatives pricing problem analytically. Let $\tilde{Y}$ be a diffusion process defined as $d\tilde{Y} = mt + s\,|1 + dU|\,dZ$. The problem arising from the possibility of having a negative volatility in (1) is resolved by Proposition 1. Without loss of generality let us parameterise (2) by letting $h = \sqrt{1-g^2}$.

Proposition 1
The distribution of process Y is the same as one of $\tilde{Y}$.

Proposition 2
If $u_0 \sim N(0,1)$ and u_t is a stationary process, i.e. $|g| < 1$, then the characteristic function of y_t is

$$\mathrm{E}[\exp(-q\,y)] = \frac{1}{\sqrt{1+d^2 s^2 q^2}} \times \exp\left(-mq + \frac{s^2 q^2}{2\left(1+d^2 s^2 q^2\right)}\right).$$

Corollary

$$sd[y] = s\sqrt{1+d^2},$$
$$kur[y] = \frac{6(2d^2 + d^4)}{(1+d^2)^2},$$
$$\mathrm{cov}[y_t^2 y_{t-s}^2] = s^2[1 + 2d^2(1+2a^s) + d^4(1+2a^{2s})].$$

Due to the parameterisation of u_t the marginal distribution of y_t is invariant to g. In fact, the probability density function (pdf) of y_t is

$$f(y) = \frac{1}{pds} e^{-\frac{1}{2d^2}} K_0\left(\frac{y}{ds}\right) + G(y), \qquad (3)$$

where K_0 is a Bessel function (see Chapter 8.43 in Gradshteyn and Ryzhik(1980)) and

$$G(y) = \frac{1}{p}\int_0^{\infty} \frac{\cos(qy)}{\sqrt{1+d^2 s^2 q^2}} e^{-\frac{1}{2d^2}} \left(e^{-\frac{1}{2d^2(1+d^2 s^2 q^2)}} - 1 \right) dq \;.$$

From the expansion of K_0 into the sum of an infinite series (see for example section 8.447 in Gradshteyn and Ryzhik(1980)), if $y \to 0$ then

$$K_0\left(\frac{y}{ds}\right) \sim \log\left(\frac{ds}{y}\right).$$

Since $\lim_{y \to 0} G(y) < \infty$, the pdf $f(y)$ exhibits a singularity point at zero. Regardless of the singularity, an approximation to the distribution of y_t is obtained by using Fast Fourier Transform (FFT), turns out to be consistent with the one derived from the numerical integration of $f(y)$ in (3).

The real data analysis is provided for the daily British Pound/US Dollar exchange rate from 1989 to 1993. The Generalized Method of Moments (Hansen, 1982) is applied to estimate the parameters as $m = 0$, $s = 0.690$, $d = 0.423$ and $g = 0.857$. These estimates were derived experimenting with a number of lags in the covariance given in Corollary above, and the second and fourth moments of the y_t were included as well. The least recommended number of lags to be included is 3. The goodness-of-fit for this model is assessed by employing the χ^2 test based on the numerical inversion of the characteristic function and the FFT method. Implying the estimated parameters, the simulation was used to obtain the mesh points for the test. The observed values of the χ^2 statistic with 47 degrees of freedom are $c^2_{num} = 45.2$ and $c^2_{FFT} = 41.2$ respectively to the method used.

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RESUME

Nous nous intéressons, dans le cadre des modèles à volatilité stochastique au temps discrète à l'estimation des paramètres. Appliqué aux data réel les paramètres inconnues était estimée et deux mèthods numériques pour inversion de la caractéristique fonction sont comparée .