Stable Models for the Distribution of Equity Capital

Robert Fernholz
INTECH
1 Palmer Square
Princeton, NJ 08542, USA
bob@enhanced.com

1. Introduction

The distribution of equity capital, or equivalently, the distribution of the size of firms, has been studied extensively, using many methodologies. Historically, the goal of the research in this area has been to find plausible mechanisms by which various models of corporate behavior would result in the observed distribution of firm size. In contrast to the historical literature, our goal in this paper is to construct an equity market model that is compatible with the observed capital distribution, without regard to the process which may have produced that distribution.

2. Definitions

A stock $X$ is a positive-valued continuous semimartingale that represents the total capitalization of a company at a given time. A market $\mathcal{M}$ is a family of $n$ stocks $X_1, \ldots, X_n$, and the market weights $\mu_1, \ldots, \mu_n$ are defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad t \in [0, \infty).$$

The $k$-th rank process of the market weights $\{\mu_1, \ldots, \mu_n\}$ is defined by

$$\mu_{(k)}(t) = \max_{i_1 < \cdots < i_k} \min(\mu_{i_1}(t), \ldots, \mu_{i_k}(t)), \quad t \in [0, \infty),$$

The capital distribution is $(\mu_{(1)}(t), \ldots, \mu_{(n)}(t))$. The capital distribution curve is the plot of log $\mu_{(k)}(t)$ versus log $k$. The capital distribution follows a Pareto distribution if the capital distribution curve is approximately a straight line. Empirically, capital distribution curves for equity markets are usually somewhat concave.

For $t \in [0, \infty)$, let $p_t$ be the random permutation of $\{1, \ldots, n\}$ such that for $k$ in $\{1, \ldots, n\}$,

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \quad \text{and} \quad p_t(k) < p_t(k + 1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t).$$

For a continuous semimartingale $X$, the local time $\Lambda_X$ is the process defined for $t \in [0, \infty)$ by

$$\Lambda_X(t) = \frac{1}{2} \left( |X(t)| - |X(0)| - \int_0^t \text{sgn}(X(s)) \, dX(s) \right).$$
3. Stable Models for the Capital Distribution

**Definition.** The capital distribution \((\mu_1, \ldots, \mu_n)\) of the market \(M\) is asymptotically stable if:

i) \(\lim_{t \to \infty} t^{-1} \log \mu_n(t) = 0, \quad \text{a.s.}\)

ii) for \(k = 1, \ldots, n-1, \quad \lim_{t \to \infty} t^{-1} \log \mu_n(t) = \lambda_{k,k+1} > 0, \quad \text{a.s.}\)

iii) for \(k = 1, \ldots, n-1, \quad \lim_{t \to \infty} t^{-1} (\log \mu_k(t) - \log \mu_{k+1}(t)) = \sigma_{k,k+1}^2 > 0, \quad \text{a.s.}\)

A *stable model* for the differences between consecutive ranked weights is defined by

\[
d(\log \mu_k(t) - \log \mu_{k+1}(t)) = -\lambda_{k,k+1} \, dt + d\Lambda_{\log \mu_k, \log \mu_{k+1}}(t) + dM_{k,k+1}(t),
\]

for \(k = 1, \ldots, n-1\) and \(t \in [0, \infty)\). Here the \(M_{k,k+1}\) are continuous martingales with \(M_{k,k+1}(t) = \sigma_{k,k+1}^2 t\), for \(t \in [0, \infty)\), a.s. With the stable model, the slope of the capital distribution curve at the point above \(\log k\) can be estimated by

\[
\frac{1}{T} \int_0^T \frac{\log \mu_k(t) - \log \mu_{k+1}(t)}{\log(k) - \log(k+1)} \, dt \approx \frac{k \sigma_{k,k+1}^2}{2 \lambda_{k,k+1}},
\]

for \(k = 1, \ldots, n-1\).

For an asymptotically stable capital distribution, define

\[
g_k = \frac{1}{2} \lambda_{k-1,k} - \frac{1}{2} \lambda_{k,k+1}, \quad \text{for} \ k = 1, \ldots, n,
\]

and

\[
\sigma_k^2 = \frac{1}{4} (\sigma_{k-1,k}^2 + \sigma_{k,k+1}^2), \quad \text{for} \ k = 2, \ldots, n-1,
\]

\[
\sigma_1^2 = \frac{1}{2} \sigma_{1,2}^2, \quad \text{and} \quad \sigma_n^2 = \frac{1}{2} \sigma_{n-1,n}^2.
\]

Then the first-order model for the market is given by

\[
d \log X_i(t) = g_{q_i(t)} \, dt + \sigma_{q_i(t)} \, dV_i(t), \quad t \in [0, \infty),
\]

for \(i = 1, \ldots, n\), where \(q_i\) is the inverse of the permutation \(p_i\), and \((V_1, \ldots, V_n)\) is an \(n\)-dimensional Brownian motion. Empirical tests indicate that the first-order model accurately reflects the short-term behavior of the capital distribution in the U.S. equity market.

**REFERENCES**


**RESUME**

On présente une modèle stable pour la distribution du capital d’un marché d’actions. Ce modèle est compatible avec la distribution observée du capital.