Stable Models for the Distribution of Equity Capital

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1. Introduction

The distribution of equity capital, or equivalently, the distribution of the size of firms, has been studied extensively, using many methodologies. Historically, the goal of the research in this area has been to find plausible mechanisms by which various models of corporate behavior would result in the observed distribution of firm size. In contrast to the historical literature, our goal in this paper is to construct an equity market model that is compatible with the observed capital distribution, without regard to the process which may have produced that distribution.

2. Definitions

A *stock* X is a positive-valued continuous semimartingale that represents the total capitalization of a company at a given time. A *market* $\mathcal{M}$ is a family of n stocks $X_1, \ldots, X_n$, and the *market weights* $\mu_1, \ldots, \mu_n$ are defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad t \in [0, \infty).$$

The k-th *rank process* of the market weights $\{\mu_1, \ldots, \mu_n\}$ is defined by

$$\mu_{(k)}(t) = \max_{i_1 < \cdots < i_k} \min\big(\mu_{i_1}(t), \ldots, \mu_{i_k}(t)\big), \quad t \in [0, \infty),$$

The *capital distribution* is $(\mu_{(1)}(t), \ldots, \mu_{(n)}(t))$. The *capital distribution curve* is the plot of $\log \mu_{(k)}(t)$ versus $\log k$. The capital distribution follows a *Pareto distribution* if the capital distribution curve is approximately a straight line. Empirically, capital distribution curves for equity markets are usually somewhat concave.

For $t \in [0, \infty)$, let p_t be the random permutation of $\{1, \ldots, n\}$ such that for k in $\{1, \ldots, n\}$,

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \quad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t).$$

For a continuous semimartingale X, the *local time* Λ_X is the process defined for $t \in [0, \infty)$ by

$$\Lambda_X(t) = \frac{1}{2}\Big(|X(t)| - |X(0)| - \int_0^t \operatorname{sgn}(X(s))\, dX(s)\Big).$$

3. Stable Models for the Capital Distribution

Definition. The capital distribution $(\mu_{(1)}, \ldots, \mu_{(n)})$ of the market $\mathcal{M}$ is *asymptotically stable* if:

i) $\lim_{t\to\infty} t^{-1} \log \mu_{(n)}(t) = 0$, a.s.,

ii) for $k = 1, \ldots, n-1$, $\lim_{t\to\infty} t^{-1} \Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}(t) = \boldsymbol{\lambda}_{k,k+1} > 0$, a.s.

iii) for $k = 1, \ldots, n-1$, $\lim_{t\to\infty} t^{-1} \langle \log \mu_{(k)} - \log \mu_{(k+1)} \rangle_t = \boldsymbol{\sigma}^2_{k:k+1} > 0$, a.s.

A *stable model* for the differences between consecutive ranked weights is defined by

$$d\big(\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t)\big) = -\boldsymbol{\lambda}_{k,k+1}\, dt + d\Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}(t) + dM_{k:k+1}(t),$$

for $k = 1, \ldots, n-1$ and $t \in [0, \infty)$. Here the $M_{k:k+1}$ are continuous martingales with $\langle M_{k:k+1} \rangle_t = \boldsymbol{\sigma}^2_{k:k+1} t$, for $t \in [0, \infty)$, a.s. With the stable model, the slope of the capital distribution curve at the point above $\log k$ can be estimated by

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t)}{\log(k) - \log(k+1)} dt \approx -\frac{k\boldsymbol{\sigma}^2_{k:k+1}}{2\boldsymbol{\lambda}_{k,k+1}},$$

for $k = 1, \ldots, n-1$.

For an asymptotically stable capital distribution, define

$$\boldsymbol{g}_k = \frac{1}{2}\boldsymbol{\lambda}_{k-1,k} - \frac{1}{2}\boldsymbol{\lambda}_{k,k+1}, \quad \text{for } k = 1, \ldots, n,$$

and

$$\boldsymbol{\sigma}^2_k = \frac{1}{4}(\boldsymbol{\sigma}^2_{k-1:k} + \boldsymbol{\sigma}^2_{k:k+1}), \quad \text{for } k = 2, \ldots, n-1,$$
$$\boldsymbol{\sigma}^2_1 = \frac{1}{2}\boldsymbol{\sigma}^2_{1:2}, \quad \text{and} \quad \boldsymbol{\sigma}^2_n = \frac{1}{2}\boldsymbol{\sigma}^2_{n-1:n}.$$

Then the *first-order model* for the market is given by

$$d \log X_i(t) = \boldsymbol{g}_{q_t(i)}\, dt + \boldsymbol{\sigma}_{q_t(i)}\, dV_i(t), \quad t \in [0, \infty),$$

for $i = 1, \ldots, n$, where q_t is the inverse of the permutation p_t, and $(V_1, \ldots, V_n)$ is an n-dimensional Brownian motion. Empirical tests indicate that the first-order model accurately reflects the short-term behavior of the capital distribution in the U.S. equity market.

REFERENCES

Fernholz, R. (2001). Stable models for the distribution of equity capital. Preprint. Available at: `http://papers.ssrn.com/`

RESUME

On présente une modèle estable pour la distribution du capital d'un marché d'actions. Ce modèle est compatible avec la distribution observée du capital.