Functional Derivatives and Applications

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1. Introduction

Traditionally, functional derivatives such as Gâteaux, Hadamard, or Fréchet, have been extensively used to obtain the asymptotic distribution of the corresponding statistic. In addition, functional derivatives and von Mises expansions have been used in robustness since the early seventies as the differentiability of the corresponding functional is a first step towards a smoothness quality that is expected of robust statistical functionals. Moreover, von Mises expansions have been applied successfully in analyzing the bias of statistics. See for example Gatto and Ronchetti (1996) and Cabrera and Fernholz (1999).

Functional derivatives are also used in bootstrapping methods. When we consider the bootstrap version $T(F_n^*)$ of a statistic $T(F_n)$ to estimate its distribution, a crucial point is to determine the consistency of the bootstrap statistic. That is, we need to show that the limiting distribution of $n^{1/2}(T(F_n) - T(F))$ coincides with the limiting distribution of the bootstrap version $n^{1/2}(T(F_n^*) - T(F_n))$. This issue was addressed by Gill (1989) where he shows the consistency of the bootstrap method when the functional T is Hadamard differentiable. In other words, Gill showed that for Hadamard differentiable functionals, "the bootstrap works if the δ-method works."

In this paper we first review functional derivatives and von Mises expansions and then give several applications. In particular, we present an application of functional derivatives to a data-dependent method for correcting the content of tolerance limits. That is, we shall use the Hadamard derivative to show the consistency of a bootstrap statistic p^*, which is used to correct the p-content of tolerance intervals so that more reliable results can be obtained for a large class of population models.

2. Content Corrected Tolerance Limits

An interval $[L, U]$ is called a p-content, γ-confidence tolerance interval for a distribution function (d.f.) F if the statistics L and U, based on a random sample from F, satisfy

$$P\{F(U) - F(L) \geq p\} \geq \gamma \qquad (1)$$

where, if possible, equality replaces the last inequality. The statistics U and L are called tolerance limits.

For a normal distribution F, the sample mean $\bar{X}$ and the sample variance S^2, provide the limits $U = \bar{X} + kS$ and $L = \bar{X} - kS$ so that $[L, U]$ is a p-content, γ-confidence tolerance interval with the values of k found in Odeh and Owen (1980), Table 3. The corresponding interval $[\bar{X} - kS, \bar{X} + kS]$ is the well known **k-factor tolerance interval**. Hence, for the normal family, this tolerance interval problem is solved.

Unfortunately, if F is not normal, these k-factor limits are usually not even approximately valid. In this paper we discuss a method for finding tolerance intervals where the content p^* is data dependent, so that for the given confidence γ, the tolerance interval satisfies

$$P\{F(U) - F(L) \geq p^*\} \geq \gamma \, . \qquad (2)$$

The value p^* will be called the **corrected content**. See Fernholz and Gillespie (2001).

To find p^* for a given sample $X_1, \cdots, X_n$ from an unknown d.f. F_o, consider first the quantity

$$D_n = \sqrt{n}(F_n(\bar{X} + kS) - F_n(\bar{X} - kS) - (F_o(\bar{X} + kS) - F_o(\bar{X} - kS)))$$

and its γ-quantile d_γ. The distribution of D_n can be estimated using the bootstrap version

$$D_n^* = \sqrt{n}(F_n^*(\bar{X}^* + kS^*) - F_n^*(\bar{X}^* - kS^*) - (F_n(\bar{X}^* + kS^*) - F_n(\bar{X}^* - kS^*))),$$

so that the bootstrap γ-quantile d_γ^* can be used to find the corrected content

$$p_n^* = F_n(\bar{X} + kS) - F_n(\bar{X} - kS) - d_\gamma^*/\sqrt{n}\,.$$

In this way, when $U = \bar{X} + kS$ and $L = \bar{X} - kS$, the statement $P\{D_n \leq d_\gamma\} = \gamma$ is equivalent to (1) and will imply $P\{D_n^* \leq d_\gamma^*\} \simeq \gamma$ which is almost equivalent to (2) when the corresponding statistical functionals are consistent. The consistency holds since the functionals are Hadamard differentiable and so the asymptotic normality in this case will be preserved (see Fernholz and Gillespie (2001). This is given by the following:

Theorem. If F_o is a continuous, strictly increasing piecewise differentiable d.f. with bounded density and with finite mean and standard deviation, then the random variable D_n is asymptotically normal and the bootstrap version D_n^* is weakly consistent with D_n.

The proof of this theorem is based on the Hadamard differentiability of D_n which can be obtained through the chain rule.

Examples and applications of this corrected content method as well as the implications of correcting the content of γ-confidence tolerance intervals in practical cases will be given.

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RÉSUMÉ.

On donne des applications des dérivées fonctionelles à la méthode des limits de tolérance au contenu corrigé par le bootstrap. Les résultats asymptotiques sont obtenus par les dérivées de Hadamard. On présente aussi des exemples pratiques.