Autoregressive Hilbertian processes of order one with exogenous variables. Application.

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Let H be a real and separable Hilbert space. Let $\rho, a_1, ..., a_q$ be bounded operators on H. Let (ε_n) be a strong Hilbertian white noise. We consider the following autoregressive Hilbertian with exogenous variables of order one model, abbreviated ARHX(1):

$$X_n = \rho(X_{n-1}) + a_1(Z_{n,1}) + ... + a_q(Z_{n,q}) + \varepsilon_n \tag{1}$$

where $Z_{n,1}, ..., Z_{n,q}$ are q zero-mean autoregressive of order one - ARH(1) - exogenous variables associated respectively with operators $u_1, ..., u_q$ and strong white noises $(\eta_{n,1}), ..., (\eta_{n,q})$, i.e. $Z_{n,i} = u_i(Z_{n-1,i}) + \eta_{n,i}$. We suppose that the noises $(\varepsilon_n), (\eta_{n,1}), ..., (\eta_{n,q})$ are independant, $\sum_{n=0}^{\infty} \|\rho^n\| < \infty$, and for all $i = 1, ..., q$ $(\varepsilon_n) \amalg (Z_{n,i})$ and $\sum_{n=0}^{\infty} \|u_i^n\| < \infty$.

See Bosq (2000) for an extensive study of ARH(1) model, and Besse & Cardot (1996) for an application using smoothing splines. Such Hilbertian processes haved been studied because they can handle many continuous-time processes. ARHX(1) model takes into account explicative variable as do for example generalized additive models: cf. Hastie & Tibshirani (1990), and Davis & Speeckman (1999) for an application to air pollution forecasting.

As Mourid (1995) did for $ARH(p)$ processes, in H^{q+1} we may consider the processes $T_n = (X_n, Z_{n+1,1}, \cdots, Z_{n+1,q})^t, \varepsilon'_n = (\varepsilon_n, \eta_{n,1}, \cdots, \eta_{n,p})^t$ and

$$\rho' = \begin{pmatrix} \rho & a_1 & \cdots & \cdots & a_q \\ 0 & u_1 & 0 & \cdots & 0 \\ 0 & 0 & u_2 & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & u_q \end{pmatrix}$$

Then T is an $ARH^{q+1}(1)$ process: $T_n = \rho'(T_{n-1}) + \varepsilon'_n$.

This approach is relatively flexible: we can also consider the case where there are some kind of "feedback" properties, i.e. when causality occurs in two directions.

Under the hypothesis that $\exists j_0, \|\rho'^{j_0}\|_{\mathcal{L}} < 1$, equation (1) has a unique stationary solution of an infinite moving average type. Moreover, it is possible to write relations between the autocorrelation operators.

For limit theorems-type results, let us denote $S_n = X_1 + ... + X_n$. We can prove that

$$\frac{n^{1/4}}{(\ln n)^{\beta}} \frac{S_n}{n} \to 0 \ \text{ a.s.}, \ \beta > 1/2.$$

and a little bit more under exponential moments assumptions. Besides, if $I_{H^{q+1}} - \rho'$ is invertible in H^{q+1}

$$\frac{S_n}{\sqrt{n}} \to \mathcal{N}(0, \Gamma) \text{ in distribution,}$$

where Γ may have a simple expression in some cases.

We denote by C_n the empirical estimator of the autocovariance operator C, and by $\mathcal{S}$ the space of Hilbert-Schmidt operators on H. Under fourth moment assumptions, it is possible to get asymptotic results about $E \|C_n - C\|_{\mathcal{S}}^2$ and $\|C_n - C\|_{\mathcal{S}}$ as well as for cross-covariance operators.

For parameters estimation, we use the estimator ρ'_n of ρ' built as in Bosq (2000). Since this estimator is consistent, we extract (composing with adapted projections) from it consistent estimators of $\rho, a_1, ..., a_q, u_1, ..., u_q$.

An application of this technique to an air pollution forecasting problem show that this approach is relevant.

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RESUME

Nous donnons une reprsentation autorgressive dans un espace produit d'un processus autorgressif d'ordre un avec variables exognes autorgressives. Cette reprsentation nous permet d'obtenir l'existence et l'unicit d'une solution stationnaire, des thormes limites, et des rsultats asymptotiques sur l'autocovariance empirique. De plus, il est possible de fournir des estimateurs non-paramtriques des oprateurs dcrivant le modle. Une application la prvision de niveaux de pollution illustre cette dmarche.