Testing for Multiple Change Points for Time Series Data

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1. Introduction

Let $Y_1, Y_2, \ldots, Y_N$ be N consecutive observations from a Gaussian autoregressive process AR(1), defined as

$$Y_t = \begin{cases} c_1 + \phi_1 Y_{t-1} + \epsilon_t, & t \le t_1 \\ c_2 + \phi_2 Y_{t-1} + \epsilon_t, & t_1 < t \le t_2 \\ \vdots & \\ c_{m+1} + \phi_{m+1} Y_{t-1} + \epsilon_t, & t > t_m \end{cases} \tag{1}$$

where ε' s are *iid r.v.* $N(0, \sigma^2)$. Thus, we are interested in testing the null hypothesis of no change in the autoregressive parameters against the alternative hypothesis of m change points, that is

$$H_0 : t_1 = t_2 = \ldots = t_m = N \quad \text{against} \quad H_1 : t_1 < t_2 < \ldots < t_m < N.$$

Chen and Gupta (1995, 1996, 1997) proposed a procedure that combined the binary segmentation method given in Vostrikova (1981) with Schwarz Information Criterion (SIC) to test for multiple change points, assuming independent observations. In this paper we propose two test statistics T_1 and T_2 to test for multiple change points in the parameters when the data are correlated such as when the data arise from an autoregressive AR(1) process. The tests are based on Schwarz Information Criterion (SIC) combined with a binary segmentation procedure in order to develop an algorithm that is both fast and simple for testing several changes when the data are correlated.

2. The Proposed Test Statistics

The Schwarz Information Criterion (SIC), (Schwarz 1978), is defined as $-2\ln L(\hat{\theta}) + p\ln N$, where $L(\hat{\theta})$ is the maximum likelihood function and p is the number of free parameters. Let $SIC_{H_0}(N)$ and $SIC_{H_1}(k)$ for $k = 2, \ldots, N-1$ denote, respectively, the *SIC* under H_0 and H_1 of (1) and are given as $SIC_{H_0}(N) = (N-1)\log 2\pi + (N-1)\log \hat{\sigma}_0^2 + (N-1) + 3\log N$, where

$\hat{s}_0^{\,2} = \frac{1}{N-1}\sum_{t=2}^{N}\left(y_t - \hat{c}_1 - \hat{f}_1 y_{t-1}\right)^2$, $\hat{c}_1$ and $\hat{f}_1$ are the conditional *MLE's* of s^2, c_1 and f_1 under H_o, respectively. Similarly, $SIC_{H_1}(k) = (N-1)\log 2p + (N-1)\log \hat{s}_1^{\,2} + (N-1) + 5\log N$

where $\hat{s}_1^{\,2} = \frac{1}{N-1}\left[\sum_{t=2}^{k} (y_t - \tilde{c}_1 - \tilde{f}_1 y_{t-1})^2 + \sum_{t=k+1}^{N}(y_t - \tilde{c}_2 - \tilde{f}_2 y_{t-1})^2\right]$, $\tilde{c}_1, \tilde{f}_1, \tilde{c}_2$ and $\tilde{f}_2$

are the conditional *MLE's* of s^2, c_1, f_1, c_2 and f_2 under H_1, respectively. Let

$\Delta_N = \min_{1<k<N} \left|SIC_{H_1}(k) - SIC_{H_0}(N)\right|$, and define the proposed test statistics for H_0 against H_1

of (1) by $T_1 = \frac{a_N\left(s^{-2}\Delta_N - d_N\right)}{c_N}$, and $T_2 = a_N s^{-2} l_N^{1/2} - b_N$, where $l_N = 2\ln N - \Delta_N$,

$a_N = \sqrt{2\ln\ln N}$, $b_N = 2\ln\ln N + \ln\ln\ln N$, $c_N = b_N / a_N$, and $d_N = 2s^{-2}\ln N - c_N^2$.

Theorem 1. Let $Y_t = c_1 + f_1 Y_{t-1} + e_t$ be an AR(1) process such that e_t are *iid* $N(0, s^2)$. Then the limiting distributions of T_1 and T_2 under H_0, for all $y \in R$, are

i) $\lim_{N\to\infty} P[T_1 \le y] = 1 - \exp\left(-2e^{y/2}\right) = 1 - G(-y)$.

ii) $\lim_{N\to\infty} P[2T_2 \le y] = 1 - \exp\left(-2e^{y/2}\right) = 1 - G(-y)$,

where $G(x) = \exp\left(-2e^{-x/2}\right)$ is the Gumbel distribution with location parameter 0 and scale parameter ½

Proof. It can follow easily by applying Theorem 2.2 of Davis, Huang and Yao (1995) for $p = 1$, and the result of Horvath(1993).

Remark. The position of the change point t can be estimated by $\hat{t}$ with $\hat{t} = \arg\min_{1<k<N} SIC_{H_1}(k)$. It can be shown that $\hat{t}$ is a consistent estimator for τ.

3. An Example

Hypothetical data were generated from an AR(1) process with two change-points. The data were generated from the process that correspond to the expected values of the test statistics T_1 and T_2 with the choices of parameters as: $c_1 = 0$, $c_2 = 1.8$, $c_3 = 5.4$; $f_1 = f_2 = f_3 = -.8$; N =100. Following the steps the binary segmentation method, a change at t = 69 was detected first with the observed value of T_1 at –229.68 which was significantly smaller than the critical value –10.587 at α = 0.01. Next, the series was divided into two sub-series, one from the start till the value of the series at the detected change-point, t = 69, and the other starting immediately after the change-point till the end of the series. The algorithm then called for separate tests are performed on each of these two sub-series in the second iteration. Here, a second change was detected for the first sub-series at t = 30 with a value of T_1 = -30.417 and no change was detected for the second sub-series. The above step was repeated for each of the two sub-series in the third iteration of the procedure. Since no further change-points were detected for any of the sub-series, the algorithm was terminated.

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RESUME

We propose two test statistics for multiple change points in the parameters of an autoregressive model AR(1). The statistics are based on Schwarz Information Criterion (SIC) combined with the binary segmentation method proposed by Vostrikova (1981). The limiting null distribution of the test statistics are derived and the method is illustrated by a data set.