Kendall-type Tests in Multivariate Regression

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1. The Model and Test Statistics

Suppose $X_i$’s and $Y_j$’s are row vectors of matrices $X$ and $Y$ respectively. Let the paired data be $(X_1, Y_1), \cdots, (X_n, Y_n)$, where $X_i \in R^p$ and $Y_i \in R^q$ for $p \geq 1$ and $q \geq 1$. Then we can consider a model

\[ Y = XB/\sqrt{n} + U, \quad (1) \]

where $X$ is an $n \times p$ known matrix, $Y$ is an $n \times q$ observed matrix, and $B$ is an unknown $p \times q$ parameter matrix. And $U$ is a matrix of unobserved random disturbances whose rows for given $X$ are uncorrelated with $X$, each with mean 0 and common covariance matrix $\Sigma$. We want to test $H_0 : B/\sqrt{n} = 0$ against $H_A : B/\sqrt{n} \neq 0$. As a measure of association between $X$ and $Y$, we extend Kendall’s $\tau$ (Gibbon,1985; Choi and Marden, 1997) to a multivariate one as follows:

\[ T = \frac{1}{C(n, 2)} \sum \sum_{i<j} b(X_i - X_j)b(Y_i - Y_j)', \quad (2) \]

The statistic of our main interest based on $T$ in (2) is defined as

\[ \hat{\tau} \equiv \text{vec}(T). \quad (3) \]

For comparisons we express the linear model (1) in a vector form (Mardia et al.,1979) as follows:

\[ \text{vec}(Y) = X^*\text{vec}(B)/\sqrt{n} + \text{vec}(U), \quad (4) \]

where $X^* \equiv X \otimes I_q$. Then projection matrix $P^*$ on $X^*$ is given by $X^*(X^*X^*)^{-1}X^* = P \otimes I_q$, where $P = X(X'X)^{-1}X'$. Under the null hypothesis the conventional test statistic is

\[ LS = \text{vec}(Y)'P^*\text{vec}(Y) \sim \chi^2_{pq}. \quad (5) \]

2. Distributions and Asymptotic Relative Efficiencies

**Theorem 1** Suppose that $\Sigma_X$ and $\Sigma_Y$ are not zero, and the distribution of $X_i$’s is not degenerate. Under the null hypothesis,

\[ \sqrt{n}\hat{\tau} \rightarrow N_{pq}(0, 4\Sigma_X \otimes \Sigma_Y), \quad (6) \]
in distribution. As $n$ goes to infinity, with a consistent and equivariant estimator $\hat{\text{Cov}}(\hat{\tau})$ of $\text{Cov}(\hat{\tau})$,

$$\hat{\tau}' \hat{\text{Cov}}(\hat{\tau})^{-1} \hat{\tau} \rightarrow \chi_{pq}^2.$$  \hspace{1cm} (7)

In order to obtain the asymptotic relative efficiency (ARE) of $\hat{\tau}$ with respect to LS, we use a probability density function (7) of $X_i$’s or $Y_i$’s, which has nice spherical feature and flexibility of the tail shape (Randles, 1989).

$$f(z) = k_p \exp\left( -\frac{\nu}{2 \nu} \left| z' z / c_p \right|^\nu \right), \quad z \in \mathbb{R}^p,$$  \hspace{1cm} (8)

where

$$c_p = \nu \Gamma\left( \frac{p}{\nu} \right) / \Gamma\left( \frac{p+2}{\nu} \right), \quad k_p = \nu \Gamma\left( \frac{p}{2 \nu} \right) / \Gamma\left( \frac{p}{\nu} \right) |c_p|^{p/2}. \hspace{1cm} (9)$$

Here if $\nu > 1$, the pdf has heavy tails. If $\nu < 1$, it has light tails. If $\nu = 1$, it represents the normal distribution.

**Theorem 2** Both $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are iid from spherically symmetric distributions defined in (7) with their tail shape parameters $\nu_X$ and $\nu_Y$ respectively. Then

$$\text{ARE}(\hat{\tau}, \text{LS}) = \frac{1}{4\sigma_X \sigma_Y} \frac{(q-1)^2}{p^2 q^2} \frac{E^2[||X_1 - X_2||]}{E^2[1/||Y_1 - Y_2||]} \left( \frac{1}{||Y_1 - Y_2||} \right), \hspace{1cm} (10)$$

where $\sigma_X \equiv E\left[ (X_1 - X_2)' (X_1 - X_2) / ||X_1 - X_2|| \right] / p$. For any $p \geq 2$, $E[||X_1 - X_2||]$ is reduced to an integral of two-variable integrand. For any odd $q \geq 3$, $E[1/||Y_1 - Y_2||]$ is reduced to an integral of three-variable integrand, so that it can be calculated numerically.

**REFERENCES**


