Quality Measurement and Social Choice

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Quality is defined as preference, which means reflexive and transitive, relation. On the other hand measurement is a kind of analogy – a homomorphism between similar mathematical structures. Mathematical structure or scale is understood as a family of relations. The paper further develops Arrow' s investigations over problem of social choice. Of course there are interdependencies between individual preferences of the members of a group and that preference which governs the whole group. In the space of preferences a metric is defined and has been shown that the optimal social choice function is strictly connected with optimal approximation in this metric space.

What should be a group utility function if we know utility functions of individual members of the community? What should be a community preference with given individual preferences? In 1951 Arrow proved quite an unusual theorem which sounded like a paradox. With rather natural conditions imposed on a function assigning a group preference to individual preferences this function is a projection.

Let M stands for any finite set with $m + 1$ elements. It could be assumed that it is a set of certain goods which every community member evaluates in a given way. We make an assumption that a finite set of natural numbers $\{0, ..., n\}$ represents the whole population and that preferences P_i, $i = 0, ..., n$, are strong linear orders in the set M. They are nonreflexive, transitive and connected. A family of all preferences $O(M)$ in the set M has $(m+1)!$ elements. Every function

$$f : O^{n+1}(M) \quad O(M)$$

is called a group choice function. Obviously a function of a group choice should be invariant under community members order. It means that for every permutation s of the set $\{0, ..., n\}$ there is

$$f(P_0, ..., P_n) = f(P_{s(0)}, ..., P_{s(n)}).$$

Choice functions considered by Arrow cannot satisfy this axiom because any projection, except trivial cases, is not invariant under permutation of variables.

For any $P \in O(M)$, $(P_0, ..., P_n) \in O^{n+1}(M)$ and $(x, y) \in P$, we call the natural number

$$I(x, y) = \mathrm{card}\{i : xP_i y\}$$

an index of a pair (x, y) with reference to a configuration $(P_0, ..., P_n)$ and we will call number

$$\mathrm{ind}(P) = \underset{(x,\,y)\,\in\,P}{\quad} I(x, y)$$

an index of preference P with reference to this configuration. Function ind, with fixed configuration of individual relations $(P_0, ..., P_n)$, maps a finite set of all possible preferences $O(M)$ into a set of natural numbers $N = \{0,1, ...\}$. Therefore, there is always at least one such preference for which this function reaches maximal value.

Let f be a group choice function mapping configuration $(P_0, ..., P_n)$ into preference for which function ind, with a given configuration of individual preferences, reaches maximal value. We call it the optimal function of group choice. It is not unique. The optimal function of group choice is called a natural function if for every ordered family of preferences, when preference maximising index function belongs to it, then the function of the group choice assigns to this family a preference belonging to that configuration.

Function f is certainly invariant under permutation of arguments. It follows that the choice function f cannot be a projection and therefore, it cannot satisfy all axioms of Arrow.

Proposition. If $(a, b) \in P_i$ for $i = 1, ..., n$, then $(a, b) \in P$. It means that optimal function of group choice fulfils the first axiom of Arrow.

A group choice function f is called a semiprojection, if for every n-tuple $(P_0, ..., P_n)$ there is $f(P_0, ..., P_n) \in \{P_0, ..., P_n\}$. Every projection is certainly a semiprojection.

Configuration $(P_0, ..., P_n)$ for which the optimal relation of the group choice satisfies the condition $f(P_0, ..., P_n) \in \{P_0, ..., P_n\}$ is called the natural configuration. If $P_0 = ... = P_n$, then the configuration is natural. In that case all participants are unanimous, so the optimum group choice is at the same time the individual choice of every member of community.

Every configuration of individual preferences $(P_0, ..., P_n)$ generates some relation S. For every two elements x and y from M we have by definition xSy if, and only if $I(x, y) > (n + 1)/2$. Is relation S a preference? If it is so, then relation S maximises the index and satisfies the first Arrow's axiom. However, in a general case, S does not have to be a preference. Only with additional assumptions on individual preferences it satisfies axioms of strong linear order.

Theorem (2/3 rule). If for every two elements x, y from M there is $I(x, y) > (2/3)(n + 1)$ or $I(y, x) > (2/3)(n + 1)$, then relation S is a preference.

The 2/3 rule is the law of nature which justifies solutions adopted in the majority of countries, the solutions concern election systems and voting systems.

A family of all finite subsets $F(X)$ of the set X is a commutative ring – with unity if X is finite, when as addition we take a symmetrical difference of sets, and as multiplication the intersection. In the family $F(X)$ it is possible to introduce a norm and a distance induced by the norm. We call a number of elements of set $A \in F(X)$ a norm $|A|$ of this set. The distance $d(A, B)$ of sets A and B is a norm of a symmetrical difference of these sets. In set $O(M)$ there is a metric which is defined in this way. A distance of two preferences is an even number – a measure of incompatibility of these choices.

Lemma. *An index of relation P related to individual preference configuration $(P_0, ..., P_n)$ is given by the formula*

$$\text{ind}(P) = (m(m + 1)n - (d(P, P_0) + ... + d(P, P_n)))/2.$$

Therefore, a problem of index maximisation is equivalent to a classical problem of selection of point P minimising the sum of distances to a given n-tuple $(P_0, ..., P_n)$. Thus the choice of a group preference has been reduced to determine the centre of gravity of the individual preference configuration.

REFERENCES

Arrow, K.J. (1951). Social Choice and Individual Values. John Wiley. New York.

£yko, J., Misztal, A. and Smoluk, A. (2000). Problem of group choice and the 2/3 rule. Ekonomia Matematyczna (Mathematical Economics) 4, 21-34.

RÉSUMÉ

Dans cet article la méthode d'approximation o ptimale a été utilisée pour désigner la fonction du choix de groupe. La fonction du choix de groupe que Arrow a proposée est une projection ce qui présente un inconvénient fondamental. La semi-projection paraît une fonction du choix social la plus rationnelle. Dans cet ouvrage la relation de concordance maximale a été mise en place et les conditions où cette relation constitue une préférence ont été déterminées. Si en cas de tous les choix cette concordance se situe au niveau supérieur de 2/3, la relation de concordance maximale est une préférence optimale du groupe.