The calculation of non-centred orthant probabilities

Tetsuhisa Miwa
National Institute for Agro-Environmental Sciences, Statistics Unit
3-1-3 Kannondai
Tsukuba 305-8604, Japan
miwa@niaes.affrc.go.jp

A. J. Hayter
Georgia Institute of Technology, School of Industrial and Systems Engineering
Atlanta, Georgia 30332-0205, U.S.A.
ajh@isye.gatech.edu

Satoshi Kuriki
The Institute of Statistical Mathematics
4-6-7 Minami-Azabu, Minato-ku,
Tokyo 106-8569, Japan
kuriki@ism.ac.jp

1. Non-centred orthoscheme probabilities

We consider the non-centred orthant probability

$$P_m(\boldsymbol{\mu}, \boldsymbol{R}) = \Pr\{x_i \geq 0,\ 1 \leq i \leq m\} = \int_0^\infty \cdots \int_0^\infty \phi_m(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{R})\ dx_1 \cdots dx_m \tag{1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_m)' \sim N_m(\boldsymbol{\mu}, \boldsymbol{R})$ with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)'$ and a positive-definite correlation matrix $\boldsymbol{R} = \{\rho_{ij}\}$ with density function $\phi_m(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{R})$. If $\boldsymbol{\mu} = \boldsymbol{0} = (0, \ldots, 0)'$ and $\boldsymbol{R}$ is a tridiagonal matrix satisfying $\rho_{ij} = 0$ for $|i-j| > 1$, then $P_m(\boldsymbol{0}, \boldsymbol{R})$ is called an orthoscheme probability (Abrahamson, 1964). Miwa *et al.* (2000) give a procedure to calculate orthoscheme probabilities $P_m(\boldsymbol{0}, \boldsymbol{R})$. In this paper we present an improved procedure for evaluating any "non-centred" orthoscheme probability $P_m(\boldsymbol{\mu}, \boldsymbol{R})$.

2. Recursive integration formulae

The positive-definite tridiagonal matrix $\boldsymbol{R}$ can be decomposed as $\boldsymbol{R} = \boldsymbol{B}\boldsymbol{B}'$ where the only non-zero elements of $\boldsymbol{B} = \{b_{ij}\}$ are b_{ii}, $1 \leq i \leq m$, and $b_{i,i-1}$, $2 \leq i \leq m$. If $\boldsymbol{z} = (z_1, \ldots, z_m)' \sim N_m(\boldsymbol{0}, \boldsymbol{I}_m)$ so that $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{z} + \boldsymbol{\mu}$, then

$$x_1 = z_1 + \mu_1, \quad x_i = b_{i,i-1} z_{i-1} + b_{ii} z_i + \mu_i, \quad 2 \leq i \leq m$$

and

$$\begin{aligned} P_m(\boldsymbol{\mu}, \boldsymbol{R}) &= \Pr\{x_i \geq 0,\ 1 \leq i \leq m\} \\ &= \Pr\{z_1 + \mu_1 \geq 0,\ b_{i,i-1}z_{i-1} + b_{ii}z_i + \mu_i \geq 0,\ 2 \leq i \leq m\}. \end{aligned}$$

Now define

$$\begin{aligned} f_{m-1}(z) &= \int_{(-\mu_m - b_{m,m-1}z)/b_{mm}}^{\infty} \phi(t)dt, \\ f_{i-1}(z) &= \int_{(-\mu_i - b_{i,i-1}z)/b_{ii}}^{\infty} f_i(t)\phi(t)dt, \quad 2 \leq i \leq m-1, \end{aligned} \tag{2}$$

where $\phi(t)$ is the standard normal probability density function, so that the required probability is given by

$$P_m(\boldsymbol{\mu}, \boldsymbol{R}) = \int_{-\mu_1}^{\infty} f_1(z)\phi(z)dz. \tag{3}$$

These integral expressions can be evaluated using a recursive computational approach.

3. Comparisons with exact values

Table 1. Numerical calculations of orthoscheme probabilities

	$m=5$		$m=10$		
G	$\rho = 1/2$	$\rho = -1/2$	$\rho = 1/2$	$\rho = -1/2$	
32	0.0847221	0.001388885	0.00886321	0.2502	$\times 10^{-7}$
64	0.084722219	0.0013888888	0.0088632345	0.25051	$\times 10^{-7}$
128	0.084722222	0.0013888889	0.0088632355	0.250520	$\times 10^{-7}$
256				0.25052105	$\times 10^{-7}$
512				0.25052108	$\times 10^{-7}$
exact	0.084722222	0.0013888889	0.0088632355	0.25052108	$\times 10^{-7}$

G: number of grid points

REFERENCES

Abrahamson, I. G. (1964) Orthant probabilities for the quadrivariate normal distribution. *Ann. Math. Statist.*, **35**, 1685-1703.

Miwa, T., Hayter, A. J. and Liu, W. (2000) Calculations of level probabilities for normal random variables with unequal variances with applications to Bartholomew's test in unbalanced one-way models. *Comput. Statist. Data Anal.*, **34**, 17-32.

RESUME

Nous présentons une nouvelle procédure récursive pour calculer des probabilités noncentrées d'orthoscheme pour tous les vecteurs moyens.