Higher order large-deviation approximations in the discrete case

Kunihiko Takahashi
Institute of Mathematics, University of Tsukuba
Ibaraki 305–8571, Japan
kunihiko@math.tsukuba.ac.jp

Masafumi Akahira
Institute of Mathematics, University of Tsukuba
Ibaraki 305–8571, Japan
akahira@math.tsukuba.ac.jp

1. Introduction

Recently the higher order large-deviation approximation for the distribution of the sum of independent discrete random variables was discussed by Akahira, Takahashi and Takeuchi (1999). The approximation is closely connected with the saddlepoint approximation (see, *e.g.* Daniels (1987), Jensen (1995)). Here, a higher order large-deviation approximation to the tail probability for the distribution of the sum of independent discrete random variables is given (see also Akahira and Takahashi (2001)). A numerical comparison with others including the saddlepoint approximation is done in the binomial case. Consequently, the higher order approximation gives sufficiently accurate results.

2. The large-deviation approximation

Assume that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent integer-valued random variables and, for each $j = 1, 2, \cdots, n, \ldots$, X_j is distributed according to a probability function $p_j(x) := P\{X_j = x\}$ $(x = 0, \pm 1, \pm 2, \ldots)$. Letting $S_n := \sum_{j=1}^n X_i$, we denote a probability function of S_n by $p_n^*(y) := P\{S_n = y\}$ $(y = 0, \pm 1, \pm 2, \ldots)$. We also denote the moment generating function (m.g.f.) of X_j by $M_j(\theta) := E\left[e^{\theta X_j}\right]$ $(j = 1, 2, \ldots, n, \ldots)$, assuming that $M_j(\theta)$'s exist for values of θ in an open interval Θ which includes 0. Now, for each j, we consider a discrete exponential family $\mathcal{P}_j := \{p_{j,\theta}(x) : \theta \in \Theta\}$ of probability functions $p_{j,\theta}(x) := P_\theta\{X_j = x\} = p_j(x)e^{\theta x}M_j(\theta)^{-1}$ $(x = 0, \pm 1, \pm 2, \ldots)$, where $p_{j,0}(x) = p_j(x)$. Denote a probability function of S_n by $p_{n,\theta}^*(y) := P_\theta\{S_n = y\}$ $(y = 0, \pm 1, \pm 2, \ldots)$, for $\theta \in \Theta$, where $p_{n,0}^*(y) = p_n^*(y)$. Then we have $p_{n,\theta}^*(y) = p_n^*(y)e^{\theta y}\prod_{j=1}^n M_j(\theta)^{-1}$ and

$$p_n^*(y) = e^{-\theta y}\prod_{j=1}^n M_j(\theta) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\prod_{j=1}^n M_j(\theta + it)\prod_{j=1}^n M_j(\theta)^{-1}e^{-ity}dt.$$

Letting $K_n(\theta) := \sum_{j=1}^{n} \log M_j(\theta)$, we have

$$p_n^*(y) = e^{K_n(\theta)-\theta y} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{K_n(\theta+it)-K_n(\theta)-ity} dt.$$

Letting $K_n^{(\alpha)}(\theta) := (d^\alpha/d\theta^\alpha)K_n(\theta)$ for $\alpha = 1, 2, \ldots$. Then we consider an estimator $\hat{\theta} := \hat{\theta}(S_n)$ for θ such that $K_n^{(1)}(\hat{\theta}) = y$, where $S_n = y$ for $y = 0, \pm 1, \pm 2, \ldots$. Suppose that $K_n^{(j)}(\hat{\theta}) = O(n)$ $(j = 2, 3, \ldots)$. Then the probability function $p_n^*(y)$ of the sum S_n is asymptotically given by

$$p_n^*(y) = \frac{1}{\sqrt{2\pi K_n^{(2)}(\hat{\theta})}} e^{K_n(\hat{\theta})-\hat{\theta}y} \left[1 + \frac{K_n^{(4)}(\hat{\theta})}{8\left\{K_n^{(2)}(\hat{\theta})\right\}^2} - \frac{5\left\{K_n^{(3)}(\hat{\theta})\right\}^2}{24\left\{K_n^{(2)}(\hat{\theta})\right\}^3} + O\left(\frac{1}{n^2}\right) \right]$$

(see Akahira, Takahashi and Takeuchi (1999)).

Theorem *The upper tail probability of the distribution of S_n is given by*

$$P\{S_n \geq y\} = p_n^*(y) \sum_{k=0}^{\infty} \exp\left\{ -k\hat{\theta}_0 - \frac{k^2}{2K_n^{(2)}(\hat{\theta}_0)} - \frac{K_n^{(3)}(\hat{\theta}_0)k}{2\left\{K_n^{(2)}(\hat{\theta}_0)\right\}^2} + O\left(\frac{1}{n^2}\right) \right\} \text{ for all } y > E(S_n).$$

In a similar way we can get the lower tail probability of the distribution of S_n.

3. Numerical comparison

In the independent, identically and binomially distributed case treated in Jensen (1995), we can compare the relative errors of the Edgeworth, Lugannani-Rice (L-R) and large deviation (LD) expansion to the exact as follows.

Table The values of $P\{S_n \geq y\}$ in the case of binomial distribution $B(10, 0.15)$

y	2	3	4	5	6	7	8	9
Exact(%)	45.5700	17.9804	4.9970	0.9874	0.1383	0.0135	0.0009	0.0000
Edgeworth	0.0068	0.0008	–0.0629	0.0395	0.3525	0.2060	–0.3574	–0.7716
L-R	0.0044	0.0067	0.0080	0.0132	0.0173	0.0213	0.0381	0.0811
LD	–0.0694	–0.0216	–0.0080	–0.0034	–0.0016	–0.0011	–0.0015	–0.0061

REFERENCES

Akahira, M. and Takahashi, K. (2001). A higher order large-deviation approximation for the discrete distributions. To appear in *J. Japan Statist. Soc.*

Akahira, M., Takahashi, K. and Takeuchi, K. (1999). The higher order large-deviation approximation for the distribution of the sum of independent discrete random variables. *Commun. Statist.–Theory Meth.*, **28**(3&4), 706–726.

Daniels, H. E. (1987). Tail probability approximations. *Int. Statist. Review*, **55**, 37–48.

Jensen, J. L. (1995). *Saddlepoint Approximations.* Clarendon Press, Oxford.