Estimation for the Optimal Solution of a Quadratic Programming Problem

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1. Introduction

Suppose that we wish to estimate the optimal solution of the following quadratic programming problem:

$$\max_{\mathbf{x}} \ \pi(\mathbf{x}) = a\boldsymbol{\mu}'\mathbf{x} - \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \qquad s.t. \ \ \mathbf{A}'\mathbf{x} = \mathbf{b} \tag{1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean vector and covariance matrix of a K-variate normal distribution, and $\mathbf{A}$ is a $K \times q$ matrix of rank q and $\mathbf{b}$ is a column vector of q components. Let $\mathbf{x}^*$ be the optimal solution in (1), then

$$\mathbf{x}^* = \frac{a}{2}\left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{A}\left(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu} + \boldsymbol{\Sigma}^{-1}\mathbf{A}\left(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)^{-1}\mathbf{b} \tag{2}$$

is obtained easily by applying a Lagragian multiplier method.

For example, such a problem arises when portfolio optimization is implemented using the historical characteristics of returns on K securities assumed to be distributed as $N_K(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. In that case, the constraints of (1) is $\mathbf{1}'\mathbf{x} = 1$ where $\mathbf{1}$ is a vector of ones and $\mathbf{x}$ represents the portfolio weights allowing unrestricted short sales.

Mori (2001b) showed that an unbiased estimator of $\mathbf{x}^*$ is

$$\frac{a(n-K+q-2)}{2n}\left(\mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{S}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{S}^{-1}\right)\mathbf{m} + \mathbf{S}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{S}^{-1}\mathbf{A}\right)^{-1}\mathbf{b} \tag{3}$$

where $\mathbf{m}$ and $\mathbf{S}$ are maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ on the basis of n independent observations. Here we consider an improvement of the unbiased estimator under the loss function $\pi(\mathbf{x}^*) - \pi(\tilde{\mathbf{x}})$ where $\tilde{\mathbf{x}}$ is an estimator of $\mathbf{x}^*$.

2. Main Result

The matrix functions $\mathbf{F}_1$ and $\mathbf{F}_2$ are defined by

$$\mathbf{F}_1(\mathbf{A}, \mathbf{W}) = \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{W}^{-1}$$

$$\mathbf{F}_2(\mathbf{A}, \mathbf{W}) = \mathbf{W}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A}\right)^{-1}$$

where $\mathbf{W}$ is a $K \times K$ symmetric and nonsingular matrix and $\mathbf{A}$ is a $K \times q$ matrix. Then the optimal solution (2) can be written

$$\mathbf{x}^* = \frac{a}{2}\mathbf{F}_1(\mathbf{A}, \boldsymbol{\Sigma})\boldsymbol{\mu} + \mathbf{F}_2(\mathbf{A}, \boldsymbol{\Sigma})\mathbf{b}\,.$$

We will consider the following two estimators of $\mathbf{x}^*$ in this paper. These estimators are of the form

(1) Proportional Type

$$\tilde{\mathbf{x}}_P = \frac{ac}{2}\mathbf{F}_1(\mathbf{A}, \mathbf{S})\mathbf{m} + \mathbf{F}_2(\mathbf{A}, \mathbf{S})\mathbf{b}\,,$$

(2) Stein Type

$$\tilde{\mathbf{x}}_S = \frac{ac}{2}\left(1 - \frac{d}{\mathbf{m}'\mathbf{F}_1(\mathbf{A}, \mathbf{S})\mathbf{m}}\right)\mathbf{F}_1(\mathbf{A}, \mathbf{S})\mathbf{m} + \mathbf{F}_2(\mathbf{A}, \mathbf{S})\mathbf{b}$$

where c and d are positive constants.

The reason why we call the estimator $\tilde{\mathbf{x}}_S$ Stein type is that $\tilde{\mathbf{x}}_S$ is function of the Stein type estimator of a certain normal mean vector under the quadratic loss. Details of the derivation of $\tilde{\mathbf{x}}_S$ are given in Mori (2001a).

THEOREM 1. Let $\tilde{\mathbf{x}}_P^*$ be the proportional type estimator with $c = c^*$ where

$$c^* = \frac{(n-K+q-2)(n-K+q-4)}{n(n-3)}.$$

Then $\tilde{\mathbf{x}}_P^*$ is the best proportional type estimator in the sense that there does not exist any proportional type estimator dominating $\tilde{\mathbf{x}}_P^*$ provided $n > K - q + 4$.

THEOREM 2. If $n > K - q + 2$ and

$$c \geq \frac{(n-K+q-2)^2}{n(n-3)},$$

then the Stein type estimator $\tilde{\mathbf{x}}_S$ dominates the proportional type estimator $\tilde{\mathbf{x}}_P$ with same constant c provided

$$0 < d < \frac{2(K-q-2)(n-K+q-2)}{n(n-3)c}. \tag{4}$$

The proof of Theorem 1 and Theorem 2 are given in Mori (2001a).

The unbiased estimator (3) is the proportional type estimator $\tilde{\mathbf{x}}_P$ with

$$c = \frac{n-K+q-2}{n}.$$

Hence, we see that the unbiased estimator is dominated by the best proportional type estimator and the Stein type estimator with $c = n^{-1}(n-K+q-2)$ and d satisfying the condition (4). However, whether the best proportional type estimator is better than the Stein type estimator with $c = c^*$ depend on the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

References

［1］ Mori, H.（2001a）, Estimation for The Optimal Solution of A Quadratic Programming Problem. *Working Paper*.

［2］ Mori, H.(2001b), Unbiased Estimation for The Optimal Solution of A Quadratic Programming Problem. *Working Paper*.

Résumé

Soit $\mathbf{x}^*$ la solution optimale du problème de programme quadratique.

$$\max_{\mathbf{x}} \ \pi(\mathbf{x}) = a\boldsymbol{\mu}'\mathbf{x} - \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$$

$$s.t. \ \ \mathbf{A}'\mathbf{x} = \mathbf{b}$$

où $\boldsymbol{\mu}$ et $\boldsymbol{\Sigma}$ sont des paramètres inconnues de la loi normale $N_K(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ de K variables. Dans cet article, nous voulons estimer $\mathbf{x}^*$ avec une fonction de perte $\pi(\mathbf{x}^*) - \pi(\tilde{\mathbf{x}})$ où $\tilde{\mathbf{x}}$ est un estimateur de $\mathbf{x}^*$. Nous avons examiné deux estimateurs de $\boldsymbol{\mu}$, un qui est le type proportionel, et autre qui est le type Stein. Nous avons trouvé que les estimateurs considerés ici dominaient le estimateur inbiaisé qui avait été donné par Mori (2001b) sous quelque conditions.