On Option Pricing with the FATGBM Risky Asset Model

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The paradigm geometric Brownian motion (GBM) model in mathematical finance which has spawned the huge financial derivatives industry postulates that the price P_t at time t of a risky asset is $P_t = P_0 exp[\mu t + \sigma W(t)]$ where P_0 is the price at time 0, $\mu, \sigma^2 > 0$ are fixed constants and $W(t)$ is a standard Brownian motion. Then the corresponding log returns

$$X_t = log P_t - log P_{t-1} = \mu + \sigma(W(t) - W(t-1))$$

are iid Gaussian with mean μ and variance σ^2. Few models could be simpler.

In contrast, the typical log returns data shows a pronounced leptokurtic distribution (much higher peaks and heavier tails than Gaussian), and evidence of strong dependence, in particular that the absolute values and squares of the X's exhibit long range dependence (LRD).

These statistical problems of the GBM model are simply resolved in the fractal activity time geometric Brownian motion (FATGBM) model introduced by Heyde (1999). In this (subordinator) formulation it is supposed that $P_t = P_0 exp[\mu t + \sigma W(T_t)]$ where the (market) activity time $\{T_t\}$ is a positive increasing random process with stationary differences which is independent of the Brownian motion $\{W(t)\}$ and the differences of the $\{T_t\}$ process are LRD and have heavy tails. Clock time just does not correspond to activity time. Importantly, $\{W(T_t), \mathcal{F}(W(s), s \leq T_t)\}$ is a martingale. So the efficient market hypothesis is being retained and there are no arbitrage opportunities.

For the FATGBM model,

$$X_t = log P_t - log P_{t-1} = \mu + \sigma(W(T_t) - W(T_{t-1})) \stackrel{d}{=} \mu + \sigma(T_t - T_{t-1})^{1/2} W(1),$$

where $\stackrel{d}{=}$ denotes equality in distribution and the heavy tails of the X_t come from those of $\tau_t = T_t - T_{t-1}$, while LRD of $\{|X_t|\}$ and $\{{X_t}^2\}$ follows from that of $\{{\tau_t}^{1/2}\}$ and $\{\tau_t\}$ respectively. Other features of the model are conditional heteroscedasity, and leptokurtosis.

Although the activity time process $\{T_t\}$ is not observed directly it can be empirically constructed. This allows the process to be checked for scaling properties and a remarkable fact emerges. To a good degree of first approximation, the process $\{T_t - t\}$ is self-similar with index $H, \frac{1}{2} < H < 1$. That is, for positive c, and times from a day upwards, $T_{ct} - ct \stackrel{d}{=} c^H(T_t - t)$, where here $\stackrel{d}{=}$ denotes equality of finite dimensional distributions. For a discussion of the empirical evidence see Heyde and Liu (2001).

Subordinator models are also closely related to stochastic volatility models in which a stochastic differential equation is hypothesized for the volatility process $\{\sigma(t)\}$ which would be related to the activity time process $\{T_t\}$ herein by $\sigma^2 T_t = \int_0^t \sigma^2(u)du$. The FATGBM model circumvents such assumptions by using the observed self-similarity.

The FATGBM model, although an incomplete market model, does provide a convenient basis for risk calculations which require less assumptions than for stochastic volatility models. We shall illustrate by discussing the pricing of European call options where issues of pricing with equivalent martingale measures, and of assigning a risk premium factor, can be avoided.

Conditionally on a fixed activity time history $\{T_t\}$, the call price C_s at time s based on initial price P_0, strike price K and interest rate r is given by the Black-Scholes formula,

$$C_s(T_s) = P_0 \Phi\left(\frac{ln(P_0 K^{-1}) + rs + \sigma^2 T_s/2}{\sigma {T_s}^{1/2}}\right) - K e^{-rs} \Phi\left(\frac{ln(P_0 K^{-1}) + rs - \sigma^2 T_s/2}{\sigma {T_s}^{1/2}}\right).$$

Thus, the distribution of $C_s(T_s)$, as T_s varies, reflects the range of prices which may be appropriate. A useful summary statistic is the mean, $EC_s(T_s)$, which is what results from the use of the so-called minimal equivalent martingale measure (corresponding to zero risk premium). Calculations based on this give results which are qualitatively similar to those which are obtained for stochastic volatility models (e.g. Heath et al. (1999)), but with some important quantitative differences.

Now it is easily checked that $C_s(T)$ is an increasing function of T. Consequently, for conservative pricing, we can assign a bound $P(T_s < {T_\alpha}^+) = \alpha$ for specified α in the spirit of Mykland (2000). This is easy to do when specific distributional assumptions are made. Thus, if, as fits the empirical evidence well, the returns process $\{X_t\}$ has a (scaled) t-distribution with ν degrees of freedom, and $\{T_t\}$ has self-similar scaling with parameter H for $t \geq 1$, then the distribution of T_t and T^+ can be calculated. Typical estimated values of ν are in the range (3,5) and those of H are in the range (0.65, 0.85). It is noteworthy that, for such values, the conservative option price $C_s({T_\alpha}^+)$ is close to the Black-Scholes price when α is high (say 0.8), wheras $EC_s(T_s)$ may be substantially less.

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