Bayesian Inference for Multiple Switching Mean Models with ARMA Errors †

Young Sook Son
Chonnam National University, Department of Statistics
Kwangju, 500-757, Korea
ysson@chonnam.chonnam.ac.kr

Seong W. Kim
Credit Risk Team, Housing commercial Bank
Seoul, 150-758, Korea
swcava@hotmail.com

Sinsup Cho
Department of Statistics, Seoul National University
Seoul, 150-742, Korea
sinsup@stats.snu.ac.kr

1. Introduction

The model with locally constant means is more practical in many cases that the mean of time series changes slowly or abruptly as the time passes. We consider the multiple switching mean model with k change points in the mean and $ARMA(p,q)$ error structure as follows.

$$M_{k,p,q} \quad : \quad Y = X_{\underline{d}_k} \underline{\mu}_k + \underline{\varepsilon} \ ,$$

where $Y = (y_1, y_2, \ldots, y_n)'$, $\underline{\mu}_k = (\mu_0, \mu_1, \ldots, \mu_k)'$, $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)'$, $X_{\underline{d}_k}$ is is an $n \times (k+1)$ matrix with elements from the $(d_{j-1}+1)^{th}$ row to the $(d_j - d_{j-1})^{th}$ row in the j^{th} column being ones and else rows all zeroes, and $\underline{d}_k = (d_1, d_2, \ldots, d_k)$ is a vector of k change points. Since $\{\varepsilon_t\}$ follows a stationary and invertible $ARMA(p,q)$ process, $\Phi(B)\varepsilon_t = \Theta_q(B)a_t$, where $a_t \sim i.i.d\ N(0, \sigma^2)$, $(\underline{\phi}_p, \underline{\theta}_q)$ must be in the stationary and invertible region, $C_p \times C_q$.

We assume default priors without any information as follows; $\pi^N(\underline{\mu}_k, \sigma) \propto \sigma^{-1}$. $\pi_k(\underline{d}_k) = (n-k)^{-1} \cdot \{\prod_{i=1}^{k-1}(n - d_i - k + i)^{-1}\} \cdot I_D(\underline{d}_k)$, where $\underline{d}_k \in D = \{\underline{d}_k | d_{i-1}+1 \le d_i \le n-k+i-1, i = 1, 2, \ldots, k\}$, $\pi(\underline{\phi}_p, \underline{\theta}_q) = I_{C_p \times C_q}(\underline{\phi}_p, \underline{\theta}_q)/Volume(C_p \times C_q)$, and finally $\pi^N(\underline{\mu}_k, \sigma, \underline{d}_k, \underline{\phi}_p, \underline{\theta}_q) = \pi^N(\underline{\mu}_k, \sigma) \cdot \pi_k(\underline{d}_k) \cdot \pi(\underline{\phi}_p, \underline{\theta}_q)$.

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2. Model Selection by the FBF and Estimation by Gibbs Sampling

We define the following function of data Y and a constant b such as $0 < b \leq 1$,

$$B^N_{(k,p,q)(k',p',q')}(Y|b) = m^N_{(k,p,q)}(Y|b)/m^N_{(k',p',q')}(Y|b),$$

where $$m^N_{(k,p,q)}(Y|b) = 2^{-1}\pi^{-\frac{1}{2}(bn-k-1)} b^{\frac{bn}{2}} \Gamma\{\tfrac{1}{2}(bn-k-1)\} \textstyle\sum_{d_k \in D} \pi_k(\underline{d}_k)$$
$$\cdot \int_{[-1,1]^{p+q}} \frac{|V^{*^{-1}}_{p,q}|^{\frac{b}{2}} |X'_{\underline{d}_k} V^{*^{-1}}_{p,q} X_{\underline{d}_k}|^{\frac{1}{2}(bn-k-1)}}{|(X_{\underline{d}_k},Y)' V^{*^{-1}}_{p,q} (X_{\underline{d}_k},Y)|^{-\frac{1}{2}(bn-k-1)}} f(\underline{r}_p, \underline{r}_q) d(\underline{r}_p \times \underline{r}_q)$$

Here $V^*_{p,q}$ is an $n \times n$ matrix with replaced $(\underline{\phi}_p, \underline{\theta}_q)$ in $V_{p,q}$ such that $Cov(Y) = \sigma^2 V_{p,q}$ by partial autocorrelations $(\underline{r}_p, \underline{r}_q)$ following Jones(1987) and $f(\underline{r}_p, \underline{r}_q)$ is a rescaled beta density defined on $(-1,1)$. Details for closed form of $V^{-1}_{p,q}$ and its determinant are presented in Leeuw(1994). The integral is computed using Monte Carlo method by importance sampling.

The fractional Bayes factor of O'Hagan(1995) for testing the model $M_{k,p,q}$ to $M_{k',p',q'}$ is given by

$$B^F_{(k,p,q)(k',p',q')} = B^N_{(k,p,q)(k',p',q')}(Y|b=1) \cdot B^N_{(k',p',q')(k,p,q)}(Y|b).$$

Full conditional densities in the implementation of the Gibbs sampler for sampling parameters from the joint posterior are as follows :

(i) $\underline{\mu}_k \mid \sigma, \underline{d}_k, \underline{r}_p, \underline{r}_q \sim N_{k+1}(\hat{\underline{\mu}}_k, (X'_{\underline{d}_k} V^{*^{-1}}_{p,q} X_{\underline{d}_k})^{-1}\sigma^2)$, where $\hat{\underline{\mu}}_k = (X'_{\underline{d}_k} V^{*^{-1}}_{p,q} X_{\underline{d}_k})^{-1} X'_{\underline{d}_k} V^{*^{-1}}_{p,q} Y$,

(ii) $\sigma^2 \mid \underline{\mu}_k, \underline{d}_k, \underline{r}_p, \underline{r}_a \sim IG(n/2, 2/Q(\underline{\mu}_k, \underline{d}_k, \underline{\gamma}_p, \underline{\gamma}_q))$, where $IG_{(\alpha,\beta)}$ denotes the inverse gamma distribution and $Q(\underline{\mu}_k, \underline{d}_k, \underline{r}_p, \underline{r}_q) = (Y - X_{\underline{d}_k}\underline{\mu}_k)' V^{*^{-1}}_{p,q} (Y - X_{\underline{d}_k}\underline{\mu}_k)$.

(ii) $\underline{d}_k \mid \underline{\mu}_k, \sigma, \underline{r}_p, \underline{r}_q \sim \pi_k(\underline{d}_k) \cdot g(\underline{d}_k)$ where $g_{(\underline{d}_k)} = exp\{-\frac{1}{2\sigma^2} Q(\underline{\mu}_k, \underline{d}_k, \underline{r}_p, \underline{r}_q)\}$.

(iii) $\underline{r}_p, \underline{r}_q \mid \underline{\mu}_k, \sigma, \underline{d}_k \sim h(\underline{r}_p, \underline{r}_q)$, where $h(\underline{r}_p, \underline{r}_q) = |V^{*^{-1}}_{p,q}|^{\frac{1}{2}} exp\{-\frac{1}{2\sigma^2} Q(\underline{\mu}_k, \underline{d}_k, \underline{r}_p, \underline{r}_q)\} \cdot f(\underline{r}_p, \underline{r}_q)$.

Since the full conditional densities of d_k and $(\underline{r}_p, \underline{r}_q)$ are not standard forms, we run the Metropolis-Hastings(MH) algorithm. $(\underline{r}_p, \underline{r}_q)$ sampled finally is transformed to $(\underline{\phi}_p, \underline{\theta}_q)$.

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