Recurrence of transformations with continuous invariant measures

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1. Recurrence of an ergodic transformation

Recurrence is one of the topics of research in dynamical systems theory. H. Poincaré observed that the orbit of a given transformation defined on a phase space returns to a neighborhood of a starting point with probability 1 with some suitable conditions. In this paper we investigate the speed of recurrence when the error of recurrence is given. This type of problem was first studied by M. Boshernitzan and we extend some of his ideas. See Theorem **??**, Theorem **??**.

Let $\mathcal{A}$ be a σ-algebra on a set X. By a *measure preserving system* $(X, \mathcal{A}, \mu, T)$ we mean a probability space $(X, \mathcal{A}, \mu)$ together with a measure preserving (i.e., $\mu(T^{-1}A) = \mu(A)$ for $E \in \mathcal{A}$) map $T : X \to X$. The map T is usually called a transformation and is not necessarily one-to-one. By a *metric measure preserving system* $(X, \mathcal{A}, \mu, d, T)$ we mean a measure preserving dynamical system $(X, \mathcal{A}, \mu, T)$ together with a metric d on X such that the σ-algebra $\mathcal{A}$ is the Borel σ-algebra generated by the metric d. The map T is not assumed to be continuous relative to d.

Let (X, d) be a metric space and $\mathcal{B}$ be the Borel σ-algebra. For $A \subset X$, $\epsilon > 0$, let

$$H_{\alpha,\epsilon}(A) = \inf \sum \mathrm{diam}(U_i)^\alpha,$$

where the infimum is taken over all countable coverings of A by subsets U_i with diameters $\mathrm{diam}(U_i) < \epsilon$. The Hausdorff α-measure on X is defined by

$$H_\alpha(A) = \lim_{\epsilon \downarrow 0} H_{\alpha,\epsilon}(A) = \limsup_{\epsilon \downarrow 0} H_{\alpha,\epsilon}(A).$$

Then it is an outer measure. It is said to be σ-finite on A if A is a countable union of sets A_i with $H_\alpha(A_i) < \infty$. In this case, the metric space (X, d) has a countable base. There exists a unique value for α such that if $s < \alpha$ then $H_s(X) = \infty$ and if $s > \alpha$ then $H_s(X) = 0$. Such α is called the Hausdorff dimension of (X, d). For an introduction to Hausdorff measures, consult Edgar.

Boshernitzan proved the following fact.

Fact 1. *Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure preserving system. Assume that for some $\alpha > 0$, the Hausdorff α-measure H_α agrees with the measure μ on the σ-algebra $\mathcal{A}$. Then for μ-almost all $x \in X$ we have*

$$\liminf_{n\to\infty} n^\beta \cdot d(T^n x, x) \le 1, \text{ with } \beta = \frac{1}{\alpha}.$$

For $X = [0,1]$ the Lebesgue measure μ coincides with the Hausdorff 1-measure H_1 on X. Hence Boshernitzan obtained the following corollary.

Fact 2. *Let $X = [0,1]$. If $T : X \to X$ is Lebesgue measure preserving (not necessarily continuous), then, for almost every $x \in X$,*

$$\liminf_{n\to\infty} n \cdot |T^n x - x| \le 1.$$

The optimal value for the constant on the right hand side is not known. Probably the right hand side is bounded by a smaller constant depending on the transformation. See the simulations in Section 3. The generalization of Fact **??** to absolutely continuous invariant measures is proved in Theorem **??**.

Definition 3. Let (X, d) be a metric space. *The first return time* $R_k(x)$ is defined by

$$R_k(x) = \min\{s \ge 1 : d(T^s x, x) \le \frac{1}{2^k}\}.$$

The *k-th recurrence error* $\epsilon_k(x)$ is defined by

$$\epsilon_k(x) = d(T^{R_k(x)} x, x).$$

Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. The transformation T is said to be *ergodic* if $T^{-1}(A) = A$ modulo measure zero sets only when $\mu(A) = 0$ or 1. Choose $A \in \mathcal{A}$ such that $\mu(A) > 0$. Define the *first return time on A* by

$$R_A(x) = \min\{j \ge 1 : T^j x \in A\}$$

for $x \in A$. It is finite for a.e. $x \in X$. We have the following fundamental result.

Fact 4 (Kac's Lemma). *Let T be an ergodic transformation on a probability space $(X, \mathcal{A}, \mu)$. If $\mu(A) > 0$, then*

$$\int_A R_A(x)\, d\mu = 1.$$

Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure preserving system. Consider a ball A of radius $1/2^k$ centered at x_0. Then $R_A = R_k$ and in view of Kac's Lemma we expect that $R_k(x_0)$ is approximately equal to $1/\mu(A)$ in a sense. For a general introduction to ergodic theory, consult Petersen and Walters.

2. Transformations with absolutely continuous invariant density

For $X = [0,1]$ a measure μ is said to *absolutely continuous* if $d\mu = \rho(x)dx$ for an integrable function $\rho(x) \geq 0$. We call $\rho(x)$ the invariant density. In this section we extend Corollary **??** for transformations on $X = [0,1]$ with absolutely continuous invariant measures.

Lemma 5. *Let $\rho(x) > 0$ be an integrable function on $X = [0,1]$. Define $d : X \times X \to \mathbb{R}$ by*

$$d(x,y) = \left| \int_x^y \rho(t)dt \right|$$

for $x, y \in X$. Then (i) d is a metric on X, and (ii) μ coincides with the Hausdorff 1-measure H_1 on X.

Theorem 6. *Let $X = [0,1]$. If $T : X \to X$ preserves an absolutely continuous probability measure $\rho(x)dx$ with $\rho(x) > 0$, then, for almost every $x \in X$,*

$$\liminf_{n\to\infty} n \cdot |T^n x - x| \leq \frac{1}{\rho(x)}.$$

Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure preserving system. Fix $x \in X$ and choose a monotonically increasing sequence $\{n_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} n_j \cdot d(T^{n_j}x, x) = \liminf_{n\to\infty} n \cdot d(T^n x, x).$$

We may assume that $\{n_j \cdot d(T^{n_j}x, x)\}_{j=1}^\infty$ monotonically decreases to a limit and that $\{d(T^{n_j}x, x)\}_{j=1}^\infty$ monotonically decreases to 0 as $j \to \infty$. For each n_j there exists k_j such that $1/2^{k_j+1} < d(T^{n_j}x, x) \leq 1/2^{k_j}$. Since $R_{k_j}(x) = \min\{s \geq 1 : d(T^s x, x) \leq 1/2^{k_j}\}$, we see that $R_{k_j} \leq n_j$ and

$$\frac{1}{2^{k_j+1}} < d(T^{n_j}x, x) \leq d(T^{R_{k_j}}x, x) \leq \frac{1}{2^{k_j}}.$$

For $X = [0,1]$ we have the following result.

Theorem 7. *Let $X = [0,1]$ and let $\rho(x)dx$, $\rho(x) > 0$, be an absolutely continuous probability measure on X. Let $T : X \to X$ be an ergodic transformation that preserves $\rho(x)dx$. Then*

$$\liminf_{k\to\infty} R_k(x)\epsilon_k(x) \leq \frac{2}{\rho(x)}.$$

3. Transformations with singular continuous invariant density

A measure μ on $X \subset \mathbb{R}^n$ is called a singular measure if there exists a set K of Lebesgue measure zero such that $\mu(X \setminus K) = 0$. In other words, μ is supported by K.

Theorem 8. *Let $X = [0, 1]$ and let μ be a singular continuous probability measure on X. Let $T : X \to X$ be an ergodic transformation that preserves μ. Then*

$$\liminf_{k\to\infty} R_k(x)\epsilon_k(x) = 0.$$

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Let $X = [0, 1]$. If an ergodic transformation $T : X \to X$ preserves an absolutely continuous probability measure $\rho(x)dx$ with $\rho(x) > 0$, then it is shown that for almost every $x \in X$,

$$\liminf_{n\to\infty} n \cdot |T^n x - x| \leq \frac{1}{\rho(x)}.$$

Define the k-th first return time $R_k(x) = \min\{s \geq 1 : |T^s x - x| \leq 1/2^k\}$ and the k-th recurrence error by $\epsilon_k(x) = |T^{R_k(x)}x - x|$. Then it is shown that

$$\liminf_{k\to\infty} R_k(x)\epsilon_k(x) \leq \frac{2}{\rho(x)}.$$

For transformations with singular continuous invariant measures the right hand side is equal to zero.