Information inequalities for the Bayes risk for a family of non-regular distributions

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1. Introduction

In the non-regular case, the Cramér-Rao type inequality was discussed by Vincze (1979), Móri (1983) and others. For a family of uniform distributions on $[\theta - (1/2), \theta + (1/2)]$, the information inequality for the Bayes risk of any estimator of θ under the quadratic loss and the uniform prior distribution on an interval $[-\tau, \tau]$ is exactly given and shown to have the sharp bound by Ohyauchi and Akahira (2000, 2001). And the lower bound for the limit infimum of the Bayes risk of any estimator of θ as $\tau \to \infty$ is attained by the mid-range, which involves the result for unbiased estimators of θ by Móri (1983).

In this paper, for a family of non-regular distributions with a location parameter θ including the uniform and truncated ones, in a similar way to Akahira (1988) the stochastic expansion of the Bayes estimator is given up to the order $o_p(n^{-1})$, and the information inequality for the Bayes risk of any estimator of θ is asymptotically obtained. The related results to the above are found in Akahira and Takeuchi (1995, 2001).

2. Information inequalities for the Bayes risk

Suppose that $X_1, X_2, \dots, X_n, \dots$ is a sequence of independent and identically distributed random variables with a density $p(x - \theta)$ w.r.t. a σ-finite measure μ, where θ is a real-valued parameter. We also assume the following conditions (A1) to (A3).

(A1) $p(x) > 0$ for $a < x < b$; $p(x) = 0$ otherwise, where a and b are constants with $a < b$.

(A2) p is twice continuously differentiable in the open interval (a, b), and

$$\lim_{x\to a+0} p(x) = \lim_{x\to b-0} p(x) = c > 0;\ \lim_{x\to b-0} p'(x) = -\lim_{x\to a+0} p'(x) = h \le 0.$$

(A3) $0 < I := -\displaystyle\int_a^b \frac{d^2 \log p(x)}{dx^2} p(x) d\mu(x) < \infty.$

Let $\underline{\theta} := \max_{1\le i\le n} X_i - b$ and $\overline{\theta} := \min_{1\le i\le n} X_i - a$, and $L(u)$ be a loss function on $\mathbf{R}^1$ which is nonnegative-valued, three times continuously differentiable and monotone increasing in $|u|$. Now, we consider a uniform distribution π on an interval $[-\tau, \tau]$ as a prior distribution of θ. Then the Bayes estimator w.r.t. L and π is the estimator $\hat{\theta}$ minimizing $\int_A^B L(\hat{\theta}-\eta)\prod_{i=1}^n p(x_i - \eta)d\eta$ for *a.a.* $x[\mu]$, where $A := \max\{-\tau, \underline{\theta}\}$, $B := \min\{\tau, \overline{\theta}\}$. And the Bayes risk of an estimator $\hat{\theta}$ of θ is defined by $r_\tau(\hat{\theta}) := (1/2\tau)\int_{-\tau}^{\tau} E_\theta\left[L(\hat{\theta}-\theta)\right] d\theta$.

Theorem *Under the conditions (A1) to (A3), the information inequality for the Bayes risk w.r.t. the quadratic loss* L *and the uniform prior* π *of any estimator* $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$ *of* θ *is given by*

$$\underline{\lim}_{\tau\to\infty}\, \underline{\lim}_{n\to\infty} n\left\{n^2 r_\tau(\hat{\theta}) - \frac{1}{2c^2}\right\} \ge -\frac{1}{2c^4}(3c^2 - 4h) - \frac{5I_0}{6c^4} =: B(c)$$

Remark The Bayes estimator $\hat{\theta}_B$ is seen to attain the lower bound $B(c)$.

Example If p is a symmetrically truncated normal density, i.e. $p(x) = ke^{-x^2/2}$ for $|x| < 1/2$, $= 0$ otherwise, where k is some constant, then $c = ke^{-1/2}$, $h = -c/2$ and $I_0 = 1 - c$, hence, from the inequality of the theorem, we have $B(c) = -(3c+2)/(2c^3) - 5(1-c)/(6c^4)$ as the lower bound which coincides with that of Ohyauchi (2001) (see also Ohyauchi and Akahira (2001)).

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