

# $\bar{X}$ Charts with Warning Probability Limits for Exponential and Laplace Processes

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## 1. Introduction

The construction of  $\bar{X}$  chart with 3-sigma action limits and 2-sigma warning limits for detecting shifts in process mean has been discussed by page (1955), and was modified and adopted as the *ISO 7873:1993* standard. The use of warning limits is to enhance the sensitivity of the  $\bar{X}$  chart in detecting small shifts of the process mean, however, at the expenses of higher risk of false alarm. The stopping rules specified in the *ISO* standard suggested that action should be taken if (i) any point falls outside the action limits (zone A), or (ii)  $K$  consecutive points fall between the upper warning and upper action limits (zone W+), or (iii)  $K$  consecutive points fall between the lower warning and lower action limits (zone W-).

The conventional  $\bar{X}$  charts with symmetric action and warning limits are constructed by placing great faith in the asymptotic property that the sample mean is approximately normally distributed, even when the sample size is as small as four or five. In this paper, we shall define the action and warning probability limits of the  $\bar{X}$  charts, for samples taken from the exponential and Laplace processes, as

$$\text{Action Probability Limits: } \left( \mathbf{m} - B_{1l} \mathbf{s} / \sqrt{n}, \mathbf{m} + B_{1u} \mathbf{s} / \sqrt{n} \right) \quad (1)$$

$$\text{Warning Probability Limits: } \left( \mathbf{m} - B_{2l} \mathbf{s} / \sqrt{n}, \mathbf{m} + B_{2u} \mathbf{s} / \sqrt{n} \right) \quad (2)$$

where  $B_{1l}, B_{1u}, B_{2l}$  and  $B_{2u}$  are factors that depend on the sample size  $n$ , the underlying distribution of the quality characteristic, and the required confidence level of the probability limits.

## 2. The values of $B_{1l}, B_{1u}, B_{2l}$ and $B_{2u}$

The values of  $B_{1l}, B_{1u}, B_{2l}$  and  $B_{2u}$  for constructing  $\bar{X}$  chart with 99.73% action probability limits and 97.50% warning probability limits are tabulated in Table 1 for samples of size  $n=2(1)(10)$  taken from the exponential and Laplace population. Values of Table 1 are obtained by using the result that (i) the sample mean of  $n$  IID  $Exp(\mathbf{q}, \mathbf{b})$  random variables is  $Gamma(\mathbf{q}, \mathbf{b}/n, n)$  distributed with location parameter  $\mathbf{q}$ , scale parameter  $\mathbf{b}/n$  and shape parameter  $n$ , and (ii) the sample mean of  $n$  IID  $Laplace(\mathbf{q}, \mathbf{b})$  random variables is distributed as the difference of two IID  $Gamma(\mathbf{b}/n, n)$  random variables. Table 1 indicates that for samples of size  $n \leq 10$ , the values of

**Table1:** Factors for constructing x-bar chart with 99.73% action probability limits and 97.50% warning probability limits.

Sample size N	Exponential population				Laplace population	
	$B_{1l}$	$B_{1u}$	$B_{2l}$	$B_{2u}$	$B_{1l} = B_{1u}$	$B_{2l} = B_{2u}$
2	1.1494	5.1067	1.0155	2.7530	3.7347	2.0565
3	1.4217	4.7317	1.1867	2.6273	3.5422	2.0284
4	1.6304	4.5042	1.2911	2.5476	3.4322	2.0126
5	1.7348	4.3476	1.3629	2.4913	3.3603	2.0026
6	1.8352	4.2313	1.4159	2.4487	3.3094	1.9957
7	1.9150	4.1407	1.4572	2.4151	3.2713	1.9906
8	1.9804	4.0674	1.4904	2.3877	3.2417	1.9868
9	2.0353	4.0066	1.5179	2.3648	3.2179	1.9838
10	2.0822	3.9551	1.5411	2.3452	3.1985	1.9814

$B_{1l}, B_{1u}$  and  $B_{2l}, B_{2u}$  are far from their corresponding values of 3.0 and 2.0 as expected under the normality assumption, even when samples are taken from the symmetric Laplace population.

### 3. The Average Run Length (ARL) and Value of $K$

The ARL of  $\bar{X}$  chart with action and warning limits under the stopping rule of ISO standard is given by

$$ARL = \frac{(1 - q_1^K)(1 - q_2^K)}{(1 - q_1)(1 - q_2) - q_1q_2(1 - q_1^{K-1})(1 - q_2^{K-1}) - p(1 - q_1^K)(1 - q_2^K)}$$

where  $p$  is the probability that a sample mean falls between the upper and lower warning limits,  $q_1$  is the probability that a sample mean falls in zone W+, and  $q_2$  is the probability that a sample mean falls in zone W-.

The  $ARL(\mathbf{d})$  values of  $\bar{X}$  chart for samples taken from a Laplace population, with population mean shifts from  $m$  to  $m \pm \mathbf{d}s$  are tabulated in Table 2 for  $n=2(1)5$ ,  $K=2,3,4,\infty$  and  $\mathbf{d} = 0.0(0.2)1.6$ , where  $K=\infty$  corresponds to the  $\bar{X}$  chart without the warning limits. Table 2 indicates that the in control ARL value of the conventional  $\bar{X}$  chart with 3-sigma control limits is too small to be acceptable in quality control applications when  $n$  is small. Follows the suggestion of ISO 7873 standard, for predetermined values of  $n$ ,  $\mathbf{d}$  and specified ranges of  $ARL(0)$ ,  $ARL(\mathbf{d})$ , we shall chose the value of  $K$  that yields the maximum  $ARL(0)/ARL(\mathbf{d})$  ratio. For example, if one wishes to detect a shift of  $\pm 1.0s$  in the mean of Laplace process using samples of size  $n=4$ , and the required ARL values are  $ARL(0) \geq 250$  and  $ARL(\mathbf{d}) \leq 10$ , then the value  $K=2$  should be chosen (since  $K=2$  gives  $ARL(0)/ARL(1.0)=52.00$ , and  $K=3$  gives a ratio of 43.62).

**Table 2:** ARL values of x-bar chart with 99.73% action limits and 97.50% warning limits

n	K	Shift in mean of size $\mathbf{d}$ (multiple of $s$ )								
		.00	.20	.40	.60	.80	1.0	1.2	1.4	1.6
2	2	263.66	215.95	131.72	70.40	36.25	18.88	10.28	6.03	3.97
	3	366.86	322.36	228.78	141.67	80.11	42.32	21.63	11.39	6.69
	4	370.29	327.93	239.00	156.11	96.24	56.51	31.29	16.68	9.37
	$\infty$	370.37	328.15	239.67	157.75	99.53	62.03	38.67	24.25	15.36
	3-sigma	100.86	89.88	66.58	44.61	28.71	18.31	11.73	7.61	5.03
3	2	263.66	187.23	88.36	38.05	17.00	8.39	4.76	3.16	2.38
	3	366.86	290.00	164.17	79.41	35.68	16.10	8.06	4.76	3.26
	4	370.29	296.68	175.75	92.62	46.24	22.18	10.90	6.07	3.90
	$\infty$	370.37	296.98	176.80	95.06	50.35	27.03	14.88	8.47	5.03
	3-sigma	125.89	102.29	62.63	34.79	19.14	10.75	6.27	3.84	2.52
4	2	263.66	162.20	62.27	23.43	9.96	5.07	3.13	2.25	1.78
	3	366.86	258.97	119.43	47.77	18.89	8.41	4.56	2.96	2.14
	4	370.29	266.45	130.88	58.04	24.88	11.01	5.63	3.41	2.32
	$\infty$	370.37	266.85	132.12	60.81	28.47	13.93	7.23	4.06	2.51
	3-sigma	146.48	107.72	55.50	26.68	13.17	6.88	3.89	2.42	1.70
5	2	263.66	141.20	46.00	15.93	6.75	3.62	2.39	1.80	1.46
	3	366.86	230.90	89.05	30.97	11.62	5.39	3.15	2.15	1.61
	4	370.29	238.93	99.61	38.47	15.01	6.66	3.63	2.32	1.66
	$\infty$	370.37	239.40	101.17	41.20	17.69	8.23	4.25	2.49	1.69
	3-sigma	163.78	108.75	48.16	20.69	9.48	4.79	2.73	1.80	1.37

### References:

Page, E.S. (1955). Control chart with warning limits. *Biometrika* **42**, pp.243-257.  
 ISO 7873 (1993). *Control charts for arithmetic average with warning limits*. International Organization for standardization, Switzerland.