Bayesian analysis of event history data

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1. Multiplicative counting processes

Event history data such as survival data, multiple event data and current status data can be effectively analyzed in the framework of counting process models. In this paper, recent results of Bayesian analysis of counting process models are presented.

Let $N(t)$ be a counting process whose compensator $\Lambda(t)$ is given by

$$\Lambda(t) = \int_0^t Y(s)dA(s) \tag{1}$$

where Y is a predictable process and A is a right continuous nondecreasing nonnegative function. Typically, Y is a censoring indicator and A is a parameter of interest. For example, let X be a survival time whose distribution function is F. For a given censoring time C, define a counting process N by $N(t) = I(\min\{X, C\} \leq t)$. Then the compensator has the form (**??**) with $Y(t) = I(\min\{X, C\} \geq t)$ and A being the cumulative hazard function of F.

For a class of priors for A, consider a class of beta processes. A beta process with parameter $(\tilde{A}(t), c(t))$, where $\tilde{A}$ is a nondecreasing nonnegative function and $c(t)$ is a positive bounded function, is defined to be a Lévy process such that

$$\begin{aligned} \mathrm{E}(\mu([0,t] \times B)) &= \int_0^t \int_B \frac{c(s)}{x}(1-x)^{c(s)-1} dx d\tilde{A}_c(s) \\ &+ \sum_{s \leq t} \int_B \frac{\Gamma(\alpha_s + \beta_s)}{\Gamma(\alpha_s)\Gamma(\beta_s)} x^{\alpha_s - 1}(1-x)^{\beta_s - 1} dx I(\Delta\tilde{A}(s) > 0) \end{aligned}$$

for all $t > 0$ and for all Borel subsets B of $[0, 1]$ where

$$\tilde{A}_c(t) = \tilde{A}(t) - \sum_{s \leq t} \Delta\tilde{A}(s),$$

$$\mu([0,t] \times B) = \sum_{s \leq t} I(\Delta A(s) \in B)$$

and $\alpha_s = c(s)\Delta\tilde{A}(s)$ and $\beta_s = c(s)(1 - \Delta\tilde{A}(s))$. Details can be found in Hjort (1990) and Kim (1999).

Let $N_1, \ldots, N_n$ be independent counting processes whose compensators Λ_i are given by $\Lambda_i(t) = \int_0^t Y_i(s)dA(s)$. The posterior distribution of A given $(N_1, \ldots, N_n)$ is presented in the next theorem. The proof can be found in Kim (1999).

Theorem 1 *A priori, let A be a beta process with parameters $(\tilde{A}, c)$. Then the posterior distribution is also a beta process with parameters*

$$\tilde{A}^p(t) = \int_0^t \frac{c(s)}{c(s) + Y.(s)} d\tilde{A}(s) + \frac{1}{c(s) + Y.(s)} dN.(s), \tag{2}$$

and

$$c^p(t) = c(t) + Y.(t), \tag{3}$$

where $N.(t) = \sum_{i=1}^n N_i(t)$ and $Y.(t) = \sum_{i=1}^n Y_i(t)$.

2. Time Inhomogeneous Markov Processes

In this section, we consider Bayesian analysis of time inhomogeneous Markov process models, which include competing risks models, illness-death models etc. Markov processes can be embedded to multivariate counting processes, and so the result in the previous section, which is valid only for univariate counting processes, cannot be applied directly.

Let $X_1, \ldots, X_n$ be independent time inhomogeneous Markov processes with state space $\{0, 1, \ldots, k\}$ having cumulative intensities $A_0, \ldots, A_k$ where

$$A_h = (A_{hj} : j \neq h).$$

A priori, assume that A_{hj} are stochastically continuous independent beta processes with mean $\tilde{A}_{hj}$ and precision parameter c_{hj}. Hjort (1990) and Kim (1999) proved that the posterior distributions of A_{hj} are again independent beta processes with mean $\tilde{A}^p_{hj}$ and precision parameter c^p_{hj} where

$$\tilde{A}^p_{hj}(t) = \int_0^t \frac{c_{hj}(s)}{c_{hj}(s) + Y_{h\cdot}(s)} d\tilde{A}_{hj} + \frac{1}{c_{hj}(s) + Y_{h\cdot}(s)} dN_{hj\cdot}(s)$$

and

$$c^p_{hj}(t) = c_{hj}(t) + Y_{h\cdot}(t)$$

where

$$N_{hji}(t) = I(X_i(t) = j, X_i(t-) = h)$$

$$Y_{hi}(t) = I(X_i(t-) = h)$$

$$N_{hj\cdot} = \sum_{i=1}^n N_{hji}$$

$$N_{h\cdot\cdot} = \sum_{j \neq h} N_{hj\cdot}$$

and

$$Y_{h\cdot} = \sum_{i=1}^{n} Y_{hi\cdot}$$

This posterior distribution is correct mathematically, but not quite right for a certain practical sense. Suppose that both $N_{hj\cdot}$ and $N_{hi\cdot}$ have jumps simultaneously at time s. Then a posteriori $\Delta A_{hj}(s)$ and $\Delta A_{hi}(s)$ are independent random variables distributed according to the beta distributions. Hence $\Delta A_{hj}(s) + \Delta_{hi}(s)$ can be larger than 1 with positive probability, which violates the condition of $\sum_{j \neq h} \Delta A_{hj}(s) \leq 1$. This example shows that the posterior distribution obtained by Hjort and Kim may have its support larger than the space of cumulative intensity functions.

To overcome this deficiency, James and Kim (2001) developed a new prior process as follows. For given finite measures $\alpha_{h0}, \ldots, \alpha_{hk}$ on $[0, \infty)$, let $\gamma_{\alpha_{hi}}$ be independent gamma processes with shape measure α_{hi} respectively. Now, define a k-dimensional Lévy process $A_h = (A_{hj}, j \neq h)$ by

$$A_{hj}(t) = \int_0^t \frac{\gamma_{\alpha_{hj}}(ds)}{\sum_{j=0}^{k} \gamma_{\alpha_{hj}}[s, \infty)}. \tag{4}$$

Inspired by the beta-neutral process developed by Lo (1993), we call A_h defined by (**??**) “the multivariate beta-neutral process” with parameters $\{\alpha_{hj}\}$. The next theorem shows that the multivariate beta-neutral process prior is conjugate.

Theorem 2 *A prior, let A_h be a multivariate beta-neutral process with parameters $\{\alpha_{hj}\}$, and let A_h be independent. Then the posterior distribution of A_h are independent multivariate beta-neutral processes with parameters $\{\alpha^p_{hj}\}$ where*

$$\alpha^p_{hj} = \alpha_{hj} + N_{hj\cdot}$$

for $j \neq h$ and

$$\alpha^p_{hh} = \alpha_{hh} + Y_{h\cdot}.$$

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