A P_λ^M-Policy for an $M/G/1$ Queueing System

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In this paper, we introduce a type of P_λ^M-policy of service for an $M/G/1$ queueing system. Server is initially idle and starts to work with service rate 1 if a customer arrives. The customers arrive according to a Poisson process of rate $\nu > 0$ and the amounts of services brought by customers are i.i.d. with distribution function G. The server, however, increases his/her service rate to $M > 1$ immediately, if the workload exceeds threshold $\lambda > 0$ over, and maintains this fast service rate until he/she becomes idle again. Then, the server restarts to work with ordinary service rate 1, if another customer arrives.

We obtain the explicit formula for the stationary distribution of the workload process of $M/G/1$ queueing system under this policy. To do this, we use the decomposition technique introduced by Lee and Ahn(1998) and adopt the level crossing argument of Brill and Posner(1977).

Let $\{X(t); t \geq 0\}$ be the workload process of $M/G/1$ queueing system with P_λ^M- policy of service. We, first, decompose $\{X(t); t \geq 0\}$ into three processes $\{X_1(t); t \geq 0\}$, $\{X_2(t); t \geq 0\}$, and $\{X_3(t); t \geq 0\}$. Process $\{X_1(t); t \geq 0\}$ is formed by separating the periods of service rate 1 from the original process and then connecting these together. Process $\{X_2(t); t \geq 0\}$ is similarly formed by separating and connecting the periods of service rate $M > 1$ from the original process. $\{X_3(t); t \geq 0\}$ is formed by connecting the rest of original process, that is, $X(3) \equiv 0$ for all $t \geq 0$.

Let $F_i(x)$ be the stationary distribution function of $\{X_i(t); t \geq 0\}$, and T_i be the length of a cycle in $\{X_i(t); t \geq 0\}$, for $i = 1, 2, 3$. Let $F(x)$ be the stationary distribution function of $\{X(t); t \geq 0\}$ and T be the length of a regeneration cycle in $\{X(t); t \geq 0\}$. Then, we can show that, for $x \geq 0$,

$$F(x) = \frac{\alpha E(T_1)}{E(T)} F_1(x) + \frac{\beta E(T_2)}{E(T)} F_2(x) + \frac{1/\nu}{E(T)},$$

where $\alpha = G(\lambda)$ and $\beta = H'(\lambda)/\nu H(\lambda)$ are the probabilities that there exist the periods of service rate 1 and M, respectively, in a cycle of $\{X(t); t \geq 0\}$, and $E[T] = \alpha E[T_1] + \beta E[T_2] + 1/\nu$.

Let $P_{0\lambda;yx}$ be the probability that the process $\{X_1(t); t \geq 0\}$ downcrosses x before touching either upper barrior λ or lower barrior 0 when the process starts at y, $0 < x, y \leq \lambda$. When $y = x$, Cohen(1978) showed

$$P_{0\lambda;xx} = 1 - \frac{H(\lambda)}{H(\lambda - x)H(x)},$$

where $H(x) = \sum_{n=0}^{\infty} \rho^n G_e^{*n}(x)$, $\rho = \nu m$, $m = \int_0^\infty x dG(x)$, and $G_e(x) = (1/m)\int_0^x (1 - G(u))du$. We observe that for $0 < y \le x \le \lambda$,

$$P_{0\lambda;yx} = \frac{H(\lambda - y)}{H(\lambda - x)} - \frac{H(\lambda)H(x-y)}{H(\lambda - x)H(x)}, \quad 0 < y \le x \le \lambda, \tag{1}$$

since $P_{0\lambda;xx} = P_{y\lambda;xx} + (1/H(x-y))P_{0\lambda;yx}$. For $0 < x < y \le \lambda$, Takacs(1967) showed

$$P_{0\lambda;yx} = H(\lambda - y)/H(\lambda - x). \tag{2}$$

Let D_{yx} be the number of downcrossings of x during a cycle of $\{X_1(t); t \ge 0\}$ when the process $\{X_1(t); t \ge 0\}$ starts at level y, $0 < x, y \le \lambda$. Then, we have

$$E[D_{yx}] = \frac{P_{0\lambda;yx}}{1 - P_{0\lambda;xx}} = \frac{H(\lambda - x)H(x)}{H(\lambda)} P_{0\lambda;yx}.$$

Let D_x be the number of downcrossings of x, $0 < x \le \lambda$, during a cycle of $\{X_1(t); t \ge 0\}$. Then,

$$E[D_x] = \int_0^x E[D_{yx}] \frac{dG(y)}{G(\lambda)} + \int_x^\lambda E[D_{yx}] \frac{dG(y)}{G(\lambda)} = \frac{H(x)}{\nu G(\lambda)} \left(\frac{H'(x)}{H(x)} - \frac{H'(\lambda)}{H(\lambda)} \right).$$

Let $f_1(x)$ be the probability density function of the stationary distribution of $X_1(t)$. Then, by the level crossing argument,

$$f_1(x) = E[D_x]/E[T_1], \quad 0 < x \le \lambda,$$

where $E[T_1]$ is obtained by recalling $\int_0^\lambda f_1(x)dx = 1$.

In a cycle of $\{X_2(t); t \ge 0\}$, by scaling time parameter we can derive that

$$E[D_x] = \begin{cases} H_{\rho'}(x) & \text{if } 0 < x \le \lambda \\ H_{\rho'}(x) - \int_\lambda^x H_{\rho'}(x-y)k(y-\lambda)dy & \text{if } x > \lambda \end{cases}$$

and

$$E[T_2] = \frac{1}{M} \frac{\lambda + E[L]}{1 - \rho'} = \frac{\lambda + E[L]}{M - \nu m},$$

where $k(l)$ is the probability density function of L, the exceeding amount of starting level over λ in a cycle of $\{X_2(t)\}$. The distribution of L can be derived from Bae et al(2001).

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