On the Permutation Test for Equality of Correlations with Some Joint Distribution

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1. Permutation test

In this paper, we discuss the permutation test for the equality of correlations, $\rho_{12} = \rho_{13}$. Sakaori and Sugiyama (2000) studied the permutation test for the equality of correlation coefficients in two bivariate populations, $\rho^{(1)} = \rho^{(2)}$. For this purpose, we have to investigate another method.

Let $\boldsymbol{X} = (X_1\ X_2\ X_3)'$ be a random vector with the mean vector $\boldsymbol{\mu} = (\mu_1\ \mu_2\ \mu_3)'$ and the covariance matrix $\boldsymbol{\Sigma}$, and σ_{ii}^2 be the variance of i-th component and ρ_{ij} be the correlation coefficient among i-th component and j-th component. We consider the test for the null hypothesis $H_0 : \rho_{12} = \rho_{13}$ virsus $H_1 : \rho_{12} < \rho_{13}$. Assume that the joint distribution of Y_1, Y_2 and Y_3 are exchangeable, such as the standerdized trivariate normal distribution, where $\boldsymbol{Y} = (Y_1\ Y_2\ Y_3)' = \boldsymbol{D}^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu})$, and $\boldsymbol{D} = \mathrm{diag}(\sigma_{11}^2\ \sigma_{22}^2\ \sigma_{33}^2)$. Note that this assumption is satisfied when the marginal distributions of Y_1, Y_2 and Y_3 are the same.

Let $\boldsymbol{x}_k = (x_{k1}\ x_{k2}\ x_{k3})'$, $k = 1, \ldots, n$ be random observations and $\bar{\boldsymbol{x}}$, s_{ii} and r_{ij} be estimators of μ, σ_{ii} and ρ_{ij}, respectively. Using the Large sample theory (Slutsky's Lemma), y_{k2} and y_{k3} are asymptotically exchangeable for each k under the null hypothesis, where $\boldsymbol{y}_k = (y_{k1},\ y_{k2},\ y_{k3})' = \hat{\boldsymbol{D}}^{-\frac{1}{2}}(\boldsymbol{y}_k - \bar{\boldsymbol{y}})$ and $\hat{\boldsymbol{D}} = \mathrm{diag}(s_{11}^2\ s_{22}^2\ s_{33}^2)$. Hence, the permutation test is asymptotically exact and unbiased.

We propose the following test statistic:

$$T_F = z_{13} - z_{12}, \tag{1}$$

where z_{1i} is the Fisher's z, namely, $z_{1i} = \frac{1}{2}\log\frac{1+r_{1i}}{1-r_{1i}}$, $i = 2, 3$. In Sakaori and Sugiyama(2000), this type of statistic is better than the statistic of the difference of the correlations because of its rapid convergence to the normal distribution. We calculate the statistics T_F of the permuted data $\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^*$, and obtain the p-value by many repetitions.

2. Numerical Studies

Since above test is applied asymptotically, it does not provide an exact significance level. Hence, we make sure the actual significance levels by some simulation studies. All simulations are carried out for 10000 times repititions and 5000 times permutations, and the sample size is 50. We generate observations from a trivariate normal distribution $N(\mathbf{0}, \mathbf{\Sigma}_1)$ in Table 1 and $N(\mathbf{0}, \mathbf{\Sigma}_2)$ in Table 2, where

$$\mathbf{\Sigma}_1 = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & .65 \\ \rho & .65 & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & .40 \\ \rho_{13} & .40 & 1 \end{pmatrix}.$$

Table 1 gives the behavior of p-value of the statistics under the null hypothesis. The correlation between X_2 and X_3 is decided so that the covariance matrices become nonsingular [N. Sugiura and H. Yoshimura (1981)]. From this table, we may see the almost exactness of this test. Table 2 shows the power of the statistics T_F. The comparison of the power with some parametric tests and robustness of the test will be given in the presentation.

Table 1. The behavior of T_F under the null hypothesis

ρ	.050	.100	.200	.300	.400	.500	.600	.700	.800	.900
$-.9$	.049	.103	.200	.298	.401	.495	.599	.704	.801	.903
0	.056	.108	.214	.313	.412	.511	.607	.706	.803	.904
.9	.044	.098	.203	.301	.398	.493	.595	.695	.796	.899

Table 2. The power of T_F

ρ_{12}	ρ_{13}	T_F	ρ_{12}	ρ_{13}	T_F	ρ_{12}	ρ_{13}	T_F	ρ_{12}	ρ_{13}	T_F	ρ_{12}	ρ_{13}	T_F
$-.9$	$-.6$	.980	$-.6$	$-.3$	.531	$-.3$	0	.414	0	.3	.419	.3	.6	.531
	$-.3$	1.00		0	.982		.3	.960		.6	.981		.9	1.00
	0	1.00		.3	1.00		.6	1.00		.9	1.00	.6	.9	.979

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RESUME

Dans cette présentation, nous énonçons au sujet de l'essai de permutation pour l'égalité des coefficients de corrélation, $\rho_{12} = \rho_{13}$. Nous donnons l'application du l'essai de permutation et quelques simulations.