Bayesian Inference under Order Restrictions for Lifetime Distributions

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1. Introduction

In this paper , we discuss the parameter estimation under order restrictions for lifetime distributions. The usual methods described in Barlow et al.(1972) under order restriction may be sometimes so restricted in the sense that ML estimates do not converge to true parameters when the restriction assumption is wrong. Therefore the usual estimators don't have consistency. We propose the following method for inference under order restrictions.

2. Modeling

Assume that the random variables $X_1, \ldots, X_{n_1}$ are n_1 independent observations from a exponential distribution with parameter λ_1 , $Y_1, \ldots, Y_{n_2}$ are n_2 independent observations from a exponential distribution with parameter λ_2 and there is the order restriction $\lambda_1 > \lambda_2$. In this case , the posterior distribution becomes

$$\pi(\lambda_1, \lambda_2 | \boldsymbol{x}, \boldsymbol{y}) \quad \propto \quad f(\boldsymbol{x}|\lambda_1) f(\boldsymbol{y}|\lambda_2) p(\lambda_1|\lambda_2) p(\lambda_2). \tag{1}$$

We consider to incorporate the order restriction into the model as the conditional prior $p(\lambda_1|\lambda_2)$. Therefore the following new densitiy function is proposed.

3. The prior density of λ_1, λ_2

The conditional prior density of λ_1 given λ_2 is

$$p(\lambda_1|\lambda_2) \quad = \quad \begin{cases} \gamma \cdot e^{\frac{\lambda_1 - \lambda_2}{\theta}} & (\lambda_1 \leq \lambda_2) \\ \gamma \cdot e^{-\gamma \frac{\lambda_1 - \lambda_2}{1 - \gamma\theta}} & (\lambda_1 > \lambda_2), \end{cases} \tag{2}$$

and the prior density of λ_2 is

$$p(\lambda_2) \quad = \quad \beta \cdot e^{-\beta\lambda_2} \qquad (\lambda_2 > 0). \tag{3}$$

The right and left tails of the distribution are changing as the value of parameter θ is varying. The larger the value of θ becomes , the more the distribution is skewed to the left, and the smaller the value of θ becomes , the more it is skewed to the right. This characteristic plays an important role in controlling the strength of the order restriction.

4. The posterior mean

When the above prior is incorporated into the model, the Bayes estimates are given by

$$\begin{aligned}
&E(\lambda_1|\boldsymbol{x},\boldsymbol{y}) = \\
&\Bigg[\frac{(n_1+1)!}{(\sum x_i + \frac{\gamma}{1-\gamma\theta})^{n_1+2}}\frac{n_2!}{(\sum y_i - \frac{\gamma}{1-\gamma\theta}+\beta)^{n_2+1}} + \sum_{k=0}^{n_2}\frac{n_2!}{k!}\frac{(n_1+k+1)!}{(\sum x_i + \sum y_i + \beta)^{n_1+k+2}} \\
&\times\left\{\frac{1}{(\sum y_i + \frac{1}{\theta}+\beta)^{n_2-k+1}} - \frac{1}{(\sum y_i - \frac{\gamma}{1-\gamma\theta}+\beta)^{n_2-k+1}}\right\}\Bigg] \Bigg/ \Bigg[\frac{n_1!}{(\sum x_i + \frac{\gamma}{1-\gamma\theta})^{n_1+1}}\frac{n_2!}{(\sum y_i - \frac{\gamma}{1-\gamma\theta}+\beta)^{n_2+1}} \\
&\quad+\sum_{k=0}^{n_2}\frac{n_2!}{k!}\frac{(n_1+k)!}{(\sum x_i + \sum y_i + \beta)^{n_1+k+1}}\left\{\frac{1}{(\sum y_i + \frac{1}{\theta}+\beta)^{n_2-k+1}} - \frac{1}{(\sum y_i - \frac{\gamma}{1-\gamma\theta}+\beta)^{n_2-k+1}}\right\}\Bigg]
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&E(\lambda_2|\boldsymbol{x},\boldsymbol{y}) \\
&\quad = \Bigg[\frac{n_1!}{(\sum x_i - \frac{1}{\theta})^{n_1+1}}\frac{(n_2+1)!}{(\sum y_i + \frac{1}{\theta})^{n_2+2}} + \sum_{k=0}^{n_1}\frac{n_1!}{k!}\frac{(n_2+k+1)!}{(\sum x_i + \sum y_i + \beta)^{n_2+k+2}} \\
&\quad\times\left\{\frac{1}{(\sum x_i + \frac{\gamma}{1-\gamma\theta})^{n_1-k+1}} - \frac{1}{(\sum x_i - \frac{1}{\theta})^{n_1-k+1}}\right\}\Bigg] \Bigg/ \Bigg[\frac{n_1!}{(\sum x_i - \frac{1}{\theta})^{n_1+1}}\frac{n_2!}{(\sum y_i - \frac{1}{\theta}+\beta)^{n_2+1}} \\
&\quad+\sum_{k=0}^{n_1}\frac{n_1!}{k!}\frac{(n_2+k)!}{(\sum x_i + \sum y_i + \beta)^{n_2+k+1}}\left\{\frac{1}{(\sum x_i + \frac{\gamma}{1-\gamma\theta})^{n_1-k+1}} - \frac{1}{(\sum x_i - \frac{1}{\theta})^{n_1-k+1}}\right\}\Bigg].
\end{aligned} \tag{5}$$

REFERENCES

Barlow , Bartholomew , Bremner , Brunk. (1972). *Statistical Inference under Order Restriction. John Wiley & Sons Ltd.*

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