Some Extensions on Noncentral Partitioned Wishart Distributed Matrices

F. ten Kleij
University of Groningen, Department of Econometrics
P.O. Box 800, 9700 AV
Groningen, The Netherlands
F.ten.Kleij@eco.rug.nl

A.G.M. Steerneman
University of Groningen, Department of Econometrics
P.O. Box 800, 9700 AV
Groningen, The Netherlands
A.G.M.Steerneman@eco.rug.nl

1. Introduction

The multivariate normal and the (noncentral) Wishart distribution are important probability distributions. Well-known characteristics of these distributions can be found in textbooks such as Muirhead (1982), Eaton (1983) and Anderson (1984). The main theorem on submatrices of partitioned Wishart distributed matrices is the following. Consider $S \sim W_k(n, \Sigma)$, Σ nonsingular, and partition S and Σ:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \text{and } \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where S_{11} and Σ_{11} are $p \times p$-matrices. Define $S_{11.2} = S_{11} - S_{12}S_{22}^{-1}S_{21}$ and $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Then

(1) $S_{11.2}$ and (S_{21}, S_{22}) are independent,
(2) $\mathcal{L}\, S_{11.2} = W_p(n-k+p, \Sigma_{11.2})$,
(3) $\mathcal{L}\, S_{22} = W_{k-p}(n, \Sigma_{22})$,
(4) $\mathcal{L}\,(S_{21}|S_{22}) = \mathcal{N}_{(k-p)\times p}(S_{22}\Sigma_{22}^{-1}\Sigma_{21}, S_{22} \otimes \Sigma_{11.2})$.

This theorem is very useful, because the distributions of many statistics, such as Hotelling's T^2, follow from it. This theorem can also be applied to obtain the expectation of the inverse of a Wishart distributed matrix after some invariance considerations.

2. Some extensions

It is tempting to explore opportunities to generalize the above-mentioned theorem in a variety of ways. First, note that whereas Σ in the definition of the Wishart distribution is allowed to be singular, this theorem restricts the covariance matrix to be nonsingular. As far as we know, a full extension of the proof of the theorem for the singular case has not been formally given yet. We will show what happens if we allow singularity and replace the usual inverse of a matrix by the Moore-Penrose inverse.

Secondly, this theorem only applies to the central Wishart distribution, an analogous result for the noncentral case is not known to us. We will make an attempt to generalize parts of this theorem to the noncentral Wishart distribution. Statement (3) of the theorem can be easily generalized, but for the other statements, we encounter some difficulties. We will discuss the problems that arise and will show what can and what cannot be done.

Finally, although great importance is attached to the multivariate normal and derived distributions, the class of spherical and elliptical distributions has received increased attention over recent years (Fang and Zhang, 1990, Fang, Kotz and Ng, 1990). Attempts have been made to generalize above-mentioned results to elliptically contoured matrix distributions (Fang and Zhang, problem 3.8, 1990) but developments in this area have not widely been explored. We will study the consequences of changing the underlying normal distribution to a distribution from the class of spherical and elliptical distributions.

This paper is explorative in nature. We want to explore the boundaries and see how far we can get.

REFERENCES

Anderson, T.W. (1984). *An Introduction to Multivariate Statistical Analysis*, 2nd edition, John Wiley and Sons, New York.

Eaton, M.L. (1983). *Multivariate Statistics–a Vector Space Approach*, John Wiley and Sons, New York.

Fang, K.T., Kotz, S. and Ng. K.W. (1990). *Symmetric Multivariate and Related Distributions*, Chapman and Hall, London.

Fang, K.T., Zhang, Y.T. (1990). *Generalized Multivariate Analysis*, Springer Verlag, Berlin.

Muirhead, R.J. (1982). *Aspects of Multivariate Statistical Theory*, John Wiley and Sons, New York.

RESUME

Cet article explore les possibilités pour généraliser a plusieurs manières le théorème principal des matrices partagées distribuées de Wishart.