A Batch Ballot Theorem and its Application to M/G/1 Type Queues

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1. Intoduction

Ballot Theorems are used extensively in a variety of applications in the area of stochastic processes. Takács, [?] [?] and [?] have illustrated application of the Ballot theorem and its generalisations.The application of the generalised ballot theorem to queueing theory leads to elegant results for the simple $M/G/1$ queue. It is thought that such results are not possible for more general $M/G/1$ type queues. We, however, derive a batch ballot theorem which can be applied to derive the first passage distribution matrix, G, for the general $M/G/1$ type queue.

2. Batch Ballot Theorem

In this section we develop a general Batch Ballot theorem. First we state the generalised Ballot theorem of Takács proved in [?].

Theorem 1 (Takács(1967)) *Let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1+n_2+\ldots+n_k = n \leq k$. Among the k cyclic permutations of $(n_1, n_2, \ldots, n_k)$ there are exactly $k-n$ for which the sum of the first s elements is less than s for all $s = 1, 2, \ldots, k$.*

Mendelson, [?], considered counting the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first r elements is less than rm for $r = 1, 2, \ldots, k$. For the extended sequence $n_1, n_2, \ldots, n_k, n_{k+1}, \ldots$, where $n_{k+r} = n_r$ for $r \geq 1$, Mendelson defined δ_r and Ψ_r by $\delta_r = 1,$ if $jm - \phi_j > rm - \phi_r$ for $j > r,$ and $= 0$ otherwise;$\Psi_r = \inf_{j \geq r}\{jm - \phi_j\}$, where $\phi_r = \sum_{j=1}^{r} n_j$. For the case where $m = 1$, Takács in [?] notes that $\Psi_{r+1} - \Psi_r = \delta_r$ and further proves that the number of cyclic permutations satisfying the criteria of Theorem 1 is equal to $\sum_{r=1}^{k} \delta_r = \sum_{r=1}^{k} (\Psi_{r+1} - \Psi_r) = k - n$. Mendelson in [?] was able to show, for $m > 1$, that $\Psi_{r+1} - \Psi_r$ may exceed unity and that $\delta_r \leq \Psi_{r+1} - \Psi_r$. As a consequence, the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first s elements is less than sm for $s = 1, 2, \ldots, k$ is bounded by $\sum_{r=1}^{k} \delta_r = \sum_{r=1}^{k} (\Psi_{r+1} - \Psi_r) \leq km - n$. In the following lemma and theorems we provide exact results for these quantities. Details of the proofs are given in [?]. Here ϕ_i and Ψ_r are as before but, for $1 \le r \le k$, redefine δ_r as

$$\delta_r = \begin{cases} m & \text{if } jm - \phi_j > rm - \phi_r + m - 1 \text{ for } j > r, \\ d & \text{if } jm - \phi_j > rm - \phi_r + d - 1 \text{ for } j > r \text{ and } um - \phi_u = rm - \phi_r + d \\ & \qquad \text{for some } u > r,\ 1 \le d \le m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2 *Let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 + n_2 + \ldots + n_k = n < km$. Then*

(a) $0 \le \delta_r = \Psi_{r+1} - \Psi_r \le m$.

(b) $\sum_{r=1}^{k} \delta_r = \sum_{r=1}^{k} (\Psi_{r+1} - \Psi_r) = km - n$.

(c) For $0 \le d \le m-1$, let C_d denote the number of cyclic permutations of $(n_1, \ldots, n_k)$ for which the sum of the first s elements is less than $sm - d$ for $1 \le s \le k$. Then $C_0 + C_1 + \ldots + C_{m-1} = km - n$.

As a consequence of the above result we have following theorem.

Theorem 3 *Let $\nu_1, \ldots, \nu_k$ be cyclically interchangeable random variables taking on nonnegative integral values. Set $N_s = \nu_1 + \ldots + \nu_s$ for $1 \le s \le k$ with $N_0 = 0$. Then, for $0 \le n \le km$,we have*

$$\sum_{d=0}^{m-1} Pr[N_s < sm - d \text{ for } 1 \le s \le k | N_k = n] = \frac{(km-n)}{k}.$$

3. Application to M/G/1 Type Queues

Queues of $M/G/1$ type arise extensively in the fields of teletraffic analysis and engineering. One of the main quantities of interest for such queues is the first passage distribution matrix G. Several methods are developed for the evaluation of G. These are mainly based upon implementing successive substitutions on a truncated form of a non-linear matrix equation ([?]). Takács, [?] and [?], applied a generalised Ballot theorem to the simple $M/G/1$ queue and obtained elegant results. It has been long thought, [?], that such results were not possible for the general $M/G/1$ type queue. We, however, apply the batch ballot theorem derived in the previous section to compute the matrix G for the general $M/G/1$ type queue. Details of this are given in [?].

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