On the cusum of squares test in time series

Siyun Park
Seoul National University, Department of Statistics
San 56-1, Shin Lim-Dong, Kwan Ak-ku
Seoul, Korea
sypark@stats.snu.ac.kr

Sangyeol Lee
Seoul National University, Department of Statistics
San 56-1, Shin Lim-Dong, Kwan Ak-ku
Seoul, Korea
sylee@stats.snu.ac.kr

1. Introduction

We consider the problem of testing for a scale change in the infinite order moving average process $X_j = \sum_{i=0}^{\infty} a_i \varepsilon_{j-i}$, where ε_j are iid r.v.'s with $E|\varepsilon_1|^\alpha < \infty$ for some $\alpha > 0$. In performing the test, a robust cusum of squares test statistic analogous to Inclán and Tiao (1994)'s statistic is considered. The cusum of squares test based on the residuals from an AR(p) model is also considered.

2. Cusum of squares tests

Assume that a sequence of positive integers $\{h_n\}$ satisfies

$$H: \ h_n \to \infty \ \text{ and } \ h_n = O(n^\rho) \ \text{ for some } \rho \in (0, 1/2). \tag{1}$$

Put

$$U_j = (1 - p_j) X_j + p_j V_j, \tag{2}$$

where p_j are iid r.v.'s with $0 \le p_j \le 1$, the contaminating process $\{V_j\}$ is a sequence of iid r.v.'s with $EV_j = 0$ and $EV_j^2 < \infty$, and $\{p_j\}$, $\{V_j\}$ and $\{X_j\}$ are all independent.

Let F denote the distribution of U_1. For $u \in (0, 1)$, let ξ_u be a number such that $F(\xi_u) = u$. Provided $U_1, \cdots, U_n$ are given, set

$$\xi_{nu} = \begin{cases} U_{(n,[nu])}, & nu \quad \text{is an integer} \\ U_{(n,[nu]+1)}, & nu \quad \text{is not an integer} \ , \end{cases} \tag{3}$$

where $U_{(n1)}, \cdots, U_{(nn)}$ denote the ordered r.v.'s of $U_1, \cdots, U_n$, and $[x]$ is the largest integer not exceeding x. Let $u < v$ be numbers in (0,1). We denote $\Psi_j^2 = U_j^2 I(\xi_{nu} \le U_j \le \xi_{nv})$, $\mu^* = n^{-1}\sum_{j=1}^n \Psi_j^2$, and $(\tau^*)^2 = \sum_{|h|\le h_n} \gamma^*(h)$, where

$$\gamma^*(h) = n^{-1}\sum_{i=1}^{n-|h|} (\Psi_i^2 - \mu^*)(\Psi_{i+|h|}^2 - \mu^*) \quad \text{for} \quad |h| < n,$$

and $\{h_n\}$ is a sequence of positive integers satisfying (1).

Theorem 1. Suppose that $E|\varepsilon_1|^\alpha < \infty$, $|a_i| \le ci^{-q}$ for some $\alpha, q > 0$ with $(\alpha \wedge 1)q > 7$, and h_n satisfies (1) with $\rho \in (0, 3/8]$. Then the cusum of squares test:

$$T_n^* := \frac{n^{1/2} \quad \mu^*}{\tau^*} \max_{1\le k\le n} \left| \frac{\sum_{j=1}^k \Psi_j^2}{\sum_{j=1}^n \Psi_j^2} - \frac{k}{n} \right| \xrightarrow{d} \sup_{0\le t\le 1} |B^o(t)| \quad \text{as} \quad n \to \infty.$$

Now we consider the stationary AR(∞) process:

$$X_j - \sum_{i=1}^\infty \beta_i X_{j-i} = \varepsilon_j. \tag{4}$$

By fitting an AR(p) model, define the least squares residuals: $\hat\varepsilon_j = X_j - \hat\beta_n' X_{j-1}$. Let

$$\tau^2 = Var(\varepsilon_1^2), \quad \hat\sigma^2 = \frac{1}{n-p}\sum_{j=p+1}^n \hat\varepsilon_j^2 \quad \text{and} \quad \hat\tau^2 = \frac{1}{n-p}\sum_{j=p+1}^n \hat\varepsilon_j^4 - \hat\sigma^2,$$

where p satisfy

$$n^{-1/2} p^2 \log n \to 0 \quad \text{and} \quad n^{7/4} p r^p \to 0 \quad \text{for all} \quad r \in (0,1). \tag{5}$$

Theorem 2. Suppose that (5) holds. Then the residual based cusum of test:

$$T_n := \max_{p+1\le k\le n} \frac{(n-p)^{1/2}\hat\sigma^2}{\hat\tau} \left| \frac{\sum_{j=p+1}^k \hat\varepsilon_j^2}{\sum_{j=p+1}^n \hat\varepsilon_j^2} - \frac{k-p}{n-p} \right| \xrightarrow{d} \sup_{0\le t\le 1} |B^o(t)|, \tag{6}$$

where B^o denotes a standard Brownian bridge.

The test stastistics are proved via simulation study to perform adequately in many situations.

REFERENCES

Inclán, C. and Tiao, G. C. (1994). Use of cumulative sums of squares for retrospective detection of changes of variances. *J. Amer. Statist. Assoc.*, **89**, 913-923.

RESUME On a reflechi sur le probleme du test concernant un changement d'echelle a l'ordre infini qui deplace le procede en moyenne $Xj = \sum_{i=0}^\infty a_i \varepsilon_{j-i}$. Dans cette equation, ε_j est une variable aleatoire identiquement distribuee de facon independante avec $E\varepsilon_1 < \infty$ pour $\alpha > 0$. Afin d'executer le test, une somme cumulative robuste de statistique du test au carre analogue a celle d'Inclan et Tiao (1994) est prise en consideration. La somme cumulative du test carre a l'appui de residuels dus d'un modele AR(p) est egalement consideree.