Empirical Bayes Tests for Lower Truncation Parameters

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Introduction and Main Results. Let $X$ denote a random variable having density function $f(x|\theta) = \frac{a(x)}{A(\theta)}$, $\theta \leq x < b \leq \infty$, (1)

where $a(x)$ is a positive, continuous function on $(0,b)$, $A(\theta) = \int_0^b a(x)dx < \infty$ for every $\theta > 0$, $\theta$ is the parameter, which is distributed according to an unknown prior distribution $G$ on $(0,b)$. Two typical examples of (1.1) are: (I) the exponential distribution with a location parameter: $f(x|\theta) = e^{-(x-\theta)}$, $x \geq \theta$, and (II) the Pareto distribution: $f(x|\theta) = \alpha \theta^{\alpha}/x^{\alpha+1}$, $x \geq \theta$.

We consider the problem of testing the hypotheses $H_0: \theta \leq \theta_0$ versus $H_1: \theta > \theta_0$, where $\theta_0$ is known and $0 < \theta_0 < b$. The loss function is $l(\theta,0) = \max\{\theta - \theta_0, 0\}$ for accepting $H_0$ and $l(\theta,1) = \max\{\theta_0 - \theta, 0\}$ for accepting $H_1$. A test $\delta(x)$ is defined to be a measurable mapping from $(0,\infty)$ into $[0,1]$ so that $\delta(x) = P\{\text{accepting } H_1 | X = x\}$. Let $f_G(x) = \int f(x|\theta)dG(\theta)$ and $\phi_G(x) = E[\theta|X = x]$. If $G$ is known, the Bayes rule $\delta_G$ can be represented as

$$
\delta_G(x) = 1 \text{ if } \phi_G(x) \geq \theta_0 \iff x \geq c_G; \quad \delta_G(x) = 0 \text{ if } \phi_G(x) < \theta_0 \iff x < c_G. \quad (2)
$$

where $c_G = \inf\{x \in (\theta_0,b) : \phi_G(x) \geq \theta_0\}$. $c_G$ is called the critical point corresponding to $G$. Let $R(G,\delta)$ denote the Bayes risk of the test $\delta$ when $G$ is the prior distribution.

Since $G$ is unknown, this testing problem is formed as a compound decision problem and the empirical Bayes approach is used. Let $X_1, X_2, \cdots, X_n$ be the observations from $n$ independent past experiences. Based on $\tilde{X}_n = (X_1, X_2, \cdots, X_n)$ and the present observation $X = x$, an empirical Bayes rule $\delta_n(X, \tilde{X}_n)$ can be constructed. The performance of $\delta_n$ is measured by $R(G,\delta_n) - R(G,\delta_G)$, where $R(G,\delta_n) = E[R(G,\delta_n|\tilde{X}_n)]$. 


From (2), a monotone empirical Bayes test (MEBT) can be constructed through estimating \( c_{G} \) by \( c_{n}(X_{1}, X_{2}, \cdots, X_{n}) \), say, and defining

\[
\delta_{n} = 1 \text{ if } x \geq c_{n}; \quad \delta_{n} = 0 \text{ if } x < c_{n}.
\]

(3)

Let \( \alpha_{G}(x) = \int_{(0, x]} dG(\theta)/A(\theta) \). For some integer \( r \), define

\[
G = \{ G : \mu_{G} \leq \mu_{0}, \sup_{0 \leq i \leq r} \sup_{\theta_{0} \leq x < b} |\alpha_{G}^{(i)}(x)| \leq B_{r}, c_{0} < c_{G} \leq \rho_{0}, \min_{x \in [c_{0}, \rho_{0}]} |w'(x)| \geq L \},
\]

(4)

where \( \mu_{0} < \infty, B_{r} < \infty, \theta_{0} < c_{0} < \rho_{0} < b, c_{0} = \frac{c_{0} + \theta_{0}}{2}, \rho_{0} = \left(2\rho_{0}\right) \wedge \frac{\theta_{0} + b}{2} \) and \( L > 0 \). Assume that \( G \) is not empty. Let \( C \) be the set of all estimators \( c_{n}^{*} \) with \( c_{n}^{*} \geq 0 \) and let \( D \) be the set of all empirical Bayes rules of type (3) with \( c_{n} = c_{n}^{*} \in C \). Our first result is the lower bound of MEBT’s over \( G \).

**Theorem 1.** For some \( l > 0 \),

\[
\inf_{\delta_{n} \in D} \sup_{G \in G} [R(G, \delta_{n}^{*}) - R(G, \delta_{G})] \geq l \cdot n^{-\frac{2r}{2r+1}}.
\]

The second result is that we construct an MEBT \( \delta_{n}^{GL} \) such that the rate in the above lower bound is achieved. Then we conclude that \( n^{-\frac{2r}{2r+1}} \) is the optimal rate of monotone empirical Bayes tests and \( \delta_{n}^{GL} \) achieves this rate. So \( \delta_{n}^{GL} \) has good performance not only for small samples but also for large samples (For small sample behavior of a monotone rule, see Van Houwelingen (1976)).

**Theorem 2.** For some \( l > 0 \),

\[
\sup_{G \in G} [R(G, \delta_{n}^{GL}) - R(G, \delta_{G})] \leq l \cdot n^{-\frac{2r}{2r+1}}
\]

For the construction of \( \delta_{n}^{GL} \) and proofs of Theorem 1 and 2, see Gupta and Li (2000).

**REFERENCES**


**RESUME**

Nous considérons le problème des essais d’une seule direction pour les paramètres de coupures plus basses par l’approche empirique de Bayes. Nous obtenons le meilleur taux d’essais de Bayes monotones empiriques et nous construisons un essai de Bayes montone empirique qui réalise le meilleur taux.