Empirical Bayes Tests for Lower Truncation Parameters

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Introduction and Main Results. Let X denote a random variable having density function

$$f(x|\theta) = a(x)/A(\theta), \qquad \theta \le x < b \le \infty, \tag{1}$$

where $a(x)$ is a positive, continuous function on $(0, b)$, $A(\theta) = \int_\theta^b a(x)dx < \infty$ for every $\theta > 0$, θ is the parameter, which is distributed according to an unknown prior distribution G on $(0, b)$. Two typical examples of (1.1) are: (I) the exponential distribution with a location parameter: $f(x|\theta) = e^{-(x-\theta)}$, $x \ge \theta$, and (II) the Pareto distribution: $f(x|\theta) = \alpha\theta^\alpha/x^{\alpha+1}$, $x \ge \theta$.

We consider the problem of testing the hypotheses $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, where θ_0 is known and $0 < \theta_0 < b$. The loss function is $l(\theta, 0) = \max\{\theta - \theta_0, 0\}$ for accepting H_0 and $l(\theta, 1) = \max\{\theta_0 - \theta, 0\}$ for accepting H_1. A test $\delta(x)$ is defined to be a measurable mapping from $(0, \infty)$ into $[0, 1]$ so that $\delta(x) = P\{$ accepting $H_1|X = x\}$. Let $f_G(x) = \int f(x|\theta)dG(\theta)$ and $\phi_G(x) = E[\theta|X = x]$. If G is known, the Bayes rule δ_G can be represented as

$$\delta_G(x) = 1 \text{ if } \phi_G(x) \ge \theta_0 \Longleftrightarrow x \ge c_G; \quad \delta_G(x) = 0 \text{ if } \phi_G(x) < \theta_0 \Longleftrightarrow x < c_G. \tag{2}$$

where $c_G = \inf\{x \in (\theta_0, b) : \phi_G(x) \ge \theta_0\}$. c_G is called the critical point corresponding to G. Let $R(G, \delta)$ denote the Bayes risk of the test δ when G is the prior distribution.

Since G is unknown, this testing problem is formed as a compound decision problem and the empirical Bayes approach is used. Let $X_1, X_2, \cdots, X_n$ be the observations from n independent past experiences. Based on $\widetilde{X}_n = (X_1, X_2, \cdots, X_n)$ and the present observation $X = x$, an empirical Bayes rule $\delta_n(X, \widetilde{X}_n)$ can be constructed. The performance of δ_n is measured by $R(G, \delta_n) - R(G, \delta_G)$, where $R(G, \delta_n) = E[R(G, \delta_n|\widetilde{X}_n)]$.

From (2), a monotone empirical Bayes test (MEBT) can be constructed through estimating c_G by $c_n(X_1, X_2, \cdots, X_n)$, say, and defining

$$\delta_n = 1 \quad \text{if} \quad x \geq c_n; \qquad \delta_n = 0 \quad \text{if} \quad x < c_n. \tag{3}$$

Let $\alpha_G(x) = \int_{(0,x]} dG(\theta)/A(\theta)$. For some integer r, define

$$\mathcal{G} = \{G : \mu_G \leq \mu_0, \sup_{0\leq i\leq r} \sup_{\frac{\theta_0}{2}<x<b} |\alpha_G^{(i)}(x)| \leq B_r, c_0 \leq c_G \leq \rho_0, \min_{x\in[\bar{c}_0,\bar{\rho}_0]} |w'(x)| \geq L\}, \tag{4}$$

where $\mu_0 < \infty$, $B_r < \infty$, $\theta_0 < c_0 < \rho_0 < b$, $\bar{c}_0 = \frac{c_0+\theta_0}{2}$, $\bar{\rho}_0 = (2\rho_0) \wedge \frac{\rho_0+b}{2}$ and $L > 0$. Assume that $\mathcal{G}$ is not empty. Let $\mathcal{C}$ be the set of all estimators c_n^* with $c_n^* \geq 0$ and let $\mathcal{D}$ be the set of all empirical Bayes rules of type (3) with $c_n = c_n^* \in \mathcal{C}$. Our first result is the lower bound of MEBT's over $\mathcal{G}$.

Theorem 1. *For some* $l > 0$, $\inf_{\delta_n^*\in\mathcal{D}} \ \sup_{G\in\mathcal{G}}[R(G, \delta_n^*) - R(G, \delta_G)] \geq l \cdot n^{-\frac{2r}{2r+1}}$.

The second result is that we construct an MEBT δ_n^{GL} such that the rate in the above lower bound is achieved. Then we conclude that $n^{-\frac{2r}{2r+1}}$ is the optimal rate of monotone empirical Bayes tests and δ_n^{GL} achieves this rate. So δ_n^{GL} has good performance not only for small samples but also for large samples (For small sample behavior of a monotone rule, see Van Houwelingen (1976)).

Theorem 2. *For some* $l > 0$, $\sup_{G\in\mathcal{G}}[R(G, \delta_n^{GL}) - R(G, \delta_G)] \leq l \cdot n^{-\frac{2r}{2r+1}}$

For the constrcution of δ_n^{GL} and proofs of Theorem 1 and 2, see Gupta and Li (2000).

REFERENCES

[1] Gupta, S. and Li, J. (2000). Optimal rate of empirical Bayes tests with for lower truncation parameters. *Technical Report # 00-07, Statistics Department, Purdue University*

[2] Van Houwelingen, J. C. (1976). Monotone empirical Bayes tests for the continuous one-parameter exponential family. *Ann. Statist.* **4**, 981-989.

RESUME

Nous considérous le problème des essais d'une seule direction pour les paramètres de coupures plus basses par l'approche empirique de Bayes. Nous obtenons le meilleur taux d'essais de Bayes monotones empiriques et nous construisons un essai de Bayes montone empirique qui réalise le meilleur taux.