Scaling windows in the Random 2-SAT Problem

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Abstract. Discrete mathematics often focuses on the study of large combinatorial structures. Random versions of these structures under appropriate distributions are discrete systems with finite but large numbers of elements. In the limit, monotone properties of these systems can and often do undergo sharp transitions. Having established the sharpness of the transition, the next step is to determine the scaling window, which measures how much one can refine the transition region. We survey these phenomena for the satisfiability of the random 2-SAT problem, based on the paper of B. Bollobás, C. Borgs, J. T. Chayes, J. H. Kim, D. B. Wilson [BBCKW01].

(*French*) Les mathématiques discrètes portent souvent principalement sur l'étude de grandes structures combinatoires. Les versions aléatoires de ces structures sous distributions appropriés sont systèms discrets avec fini, mais grands nombres d'éléments. Dans la limite, les propriétés monocordes de ces systèms peuvent et aussi subient les transition nettes. Après l'ètablissement de la netteté de la transition, il faut determiner la porte d'échelle qui measure combien on peut raffiner la region de transition. On presente ces phńomènes pour satisfiability de probléme 2-SAT, basé sur l'article de B. Bollobás, C. Borgs, J. T. Chayes, J. H. Kim, D. B. Wilson [BBCKW01].

Keywords: constraint satisfaction problem, satisfiability, sharp threshold, phase transition, finite-size scaling.

1 Introduction

The k-satisfiability (k-SAT) problem is a canonical constraint satisfaction problem in theoretical computer science. Instances of the problem are formulae in conjunctive normal form: a k-SAT formula is a conjunction of m clauses, each of which is a disjunction of k distinct literals. The k elements of each clause are chosen from among n Boolean variables and their negations. Given a formula, the decision version of the problem is whether there exists an assignment of the n variables satisfying the formula.

It is known that the k-SAT problem behaves very differently for $k = 2$ and $k \geq 3$ [Coo71]. For $k = 2$, the problem is in P [Coo71]; indeed, it can be solved by a linear time algorithm [APT79]. For $k \geq 3$, the problem is NP-complete [Coo71], so that in the worst case it is difficult to determine whether a k-SAT formula is satisfiable or not — assuming P $\neq$ NP. Note, however, that even for $k = 2$, variants of the k-SAT problem are difficult. For example, the MAX-2-SAT problem, in which one determines whether the maximum number of satisfiable clauses in a 2-SAT formula is bounded by a given integer, is an NP-complete problem [GJS76] (see also [GJ79]), and even approximating it to within a factor of $4/3 - \varepsilon$ is NP-hard [Hås97].

More recently, it has been realized that—rather than focusing on worst-case instances—it is often useful to study typical instances of the fixed-k problem as a function of the parameter $\alpha = m/n$. Consider the random k-SAT problem, in which formulae are generated by choosing uniformly at random from among all possible clauses. As m and n tend to infinity with limiting ratio $m/n \to \alpha$, considerable empirical evidence suggests that the random k-SAT problem undergoes a *phase transition* at some value $\alpha_c(k)$ of the parameter α ([MSL92], [CA93], [LT93], [KS94]): For $\alpha < \alpha_c$, a random formula is satisfiable with probability tending to one as m and n tend to infinity in the fixed ratio $\alpha = m/n$, while if $\alpha > \alpha_c$, a random formula is unsatisfiable with probability tending to one as m and n tend to infinity, again with $m/n \to \alpha$.

Existence of the phase transition is on a different footing for $k = 2$ and $k \geq 3$. For $k = 2$, it was shown by Goerdt ([Goe92], [Goe96]), Chvátal and Reed [CR92], and Fernandez de la Vega [Fer92] that a transition occurs at $\alpha_c(2) = 1$. For $k \geq 3$, it may not be possible to locate the exact value of the transition point. However, there has been considerable work bounding the value of the presumed 3-SAT threshold from below and above. Using a succession of increasingly sophisticated and clever algorithms for finding SAT solutions with high probability, lower bounds on $\alpha_c(3)$ were improved from 1 ([CF86], [CF90], [CR92]) to 1.63 [BFU93] to 3.003 [FS96] to 3.145 [Ach00] to 3.26 [AS00]. Bounding the probability of finding a solution by the expected number of solutions gave an upper bound on $\alpha_c(3)$ of 5.191 [FP83]; increasingly sophisticated counting arguments gave a succession of improved upper bounds on $\alpha_c(3)$ from 5.08 [EF95] to 4.758 [KMPS95] to 4.643 [DB97] to 4.602 [KKK96] to 4.596 [JSV00]. More recently a bound of 4.506 [DBM99] has been announced. Although these bounds are relatively tight, they nevertheless allow for the possibility of a non-sharp transition. However, motivated by the empirical evidence,

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Friedgut and later Bourgain showed that indeed there is a sharp transition [FB99] (although they did not prove that the probability of satisfiability approaches a limit). These proofs were based on a general argument which shows that global, as opposed to local, phenomena lead to sharp transitions. However, the existence of a limiting threshold is still an open problem.

Having established the sharpness of the transition, the next step is to analyze some of its properties. *Finite-size scaling* is the study of changes in the transition behavior due to finite-size effects, in particular, broadening of the transition region for finite n. To be precise, for $0 < \delta < 1$, let $\alpha_-(n,\delta)$ be the supremum over α such that for $m = \alpha n$, the probability of a random formula being satisfiable is at least $1-\delta$. Similarly, let $\alpha_+(n,\delta)$ be the infimum over α such that for $m = \alpha n$, the probability of a random formula being satisfiable is at most δ. Then, for α within the *scaling window*

$$W(n,\delta) = (\alpha_-(n,\delta), \alpha_+(n,\delta)), \tag{1}$$

the probability of a random formula being satisfiable is between δ and $1-\delta$. Since, by [FB99], for all δ, $|\alpha_+(n,\delta) - \alpha_-(n,\delta)| \to 0$ as $n \to \infty$, we say that the scaling window represents the broadening of the transition due to finite-size effects. Sometimes we shall omit the explicit δ dependence of $\alpha_\pm(n,\delta)$ and $W(n,\delta)$, writing instead $\alpha_\pm(n)$ and $W(n)$. In these cases, the power laws we quote will be uniform in δ, but the implicit constants may depend on δ.

The first model for which such broadening was established rigorously is the random graph model. The phase transition for this model, namely the sudden emergence of a giant component, was already proved by Erdős and Rényi ([ER60], [ER61]). But the characteristic width of the transition was (correctly) investigated only 24 years later by Bollobás [Bol84] (see also [Bol85] and the references therein). In particular, this work showed that the width of the scaling window $W(n)$ is $n^{-1/3+o(1)}$; the precise growth rate was later shown to be $\Theta(n^{-1/3})$ by Łuczak [Łuc90]. Many additional properties of the phase transition were then determined using generating functions [ŁPW94] [JKŁP94]. For the finite-dimensional analogue of the random graph problem, namely percolation on a low-dimensional hypercubic lattice, the broadening was established by Borgs, Chayes, Kesten and Spencer ([BCKS98a], [BCKS98b]), who also related the power law form of $\alpha_\pm(n)$ to the critical exponents of the percolation model.

The question of finite-size scaling in the k-SAT model was first addressed by Kirkpatrick and Selman [KS94], who presented both a heuristic framework and empirical evidence for analysis of the problem. There has also been subsequent empirical ([SK96], [MZKST99]) and theoretical ([MZ96], [MZ97], [MZKST99]) work, the latter using the replica method familiar from the study of disordered, frustrated models in condensed matter physics (see [MPV87] and references therein). Although the theoretical work has yielded a good deal of insight, the empirical work on finite-size scaling has been misleading [Wil00], and rigorous progress on finite-size scaling in k-SAT has been quite limited.

2 Random 2-SAT Problem

In this section, we address the question of finite-size scaling in the 2-SAT problem; in particular, the power law form of the scaling window $W(n) = (\alpha_-(n), \alpha_+(n))$, together with the rates of convergence at the boundaries of the window. Previous work on 2-SAT by Goerdt [Goe99] has shown that $\alpha_-(n) \geq 1 - O(1/\sqrt{\log n})$, while Verhoeven [Ver99] has recently obtained the result $\alpha_+(n) \leq 1 + O(n^{-1/4})$. Numerical work on the scaling window for 2-SAT is somewhat controversial: While earlier simulations [MZKST99] suggested that the window scales like $W(n) = (1 - \Theta(n^{-1/2.8}), 1 + \Theta(n^{-1/2.8}))$, recent simulations by Wilson [Wil98] indicate that the 2-SAT formulae considered in [MZKST99] are not long enough to reach the asymptotic regime.[2] Indeed, Bollobás, Borgs, Chayes, Kim and Wilson [BBCKW01] proved that $W(n) = (1 - \Theta(n^{-1/3}), 1 + \Theta(n^{-1/3}))$, as conjectured earlier by Bollobás, Borgs, Chayes and Kim [BBCK98] and predicted numerically in [Wil98]. It is also shown how the probability of satisfiability tends to 1 and 0 at the edges of the window.

In order to state the results precisely, we need a little notation. Let $x_1, \ldots, x_n$ denote n Boolean variables. Writing $\overline{x}$ for the *negation* of x, our n variables give $2n$ *literals* $x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n$. Two literals x and y are said to be *strictly distinct* if neither $x = y$ nor $x = \overline{y}$. A *k-clause* is a disjunction $C = u_1 \vee \cdots \vee u_k$ of k strictly distinct literals, and a k-SAT formula is a conjunction $F = C_1 \wedge \ldots \wedge C_m$ of k-clauses $C_1, \ldots, C_m$. We say that H is a subformula of F if it can be obtained from F by deleting some of its clauses. A k-SAT formula $F = F(x_1, \ldots, x_n)$ is said to be satisfiable, or SAT, if there exists a truth assignment $\eta_i \in \{0,1\}$, $i = 1, \ldots, n$, such that $F(\eta_1, \ldots, \eta_n) = 1$. Here, as usual, 0 stands for the logical value FALSE, and 1 is the logical value TRUE. We write "F is SAT" if the formula F is satisfiable, and "F is UNSAT" if the formula F is not satisfiable. We also sometimes use the alternative notation SAT(F) and UNSAT(F) to denote these two cases.

We consider the probability space of formulae $F_{n,m}$ chosen uniformly at random from all 2-SAT formulae with exactly m different clauses. (Here $x \vee y$ is considered to be the same as $y \vee x$, but different from e.g. $x \vee \overline{y}$.) As usual

[2] As usual, $f = \Theta(g)$ means that there exist positive, finite constants c_1 and c_2 such that $c_1 \leq f/g \leq c_2$. Unless noted otherwise, these constants are universal. In fact, in the above formulae for $W(n)$, the constants depend on δ.

in 2-SAT, it is convenient to study the phase transition in terms of the parameter ε representing the deviation of α from its critical value:

$$m = (1+\varepsilon)n. \tag{2}$$

When studying finite-size effects, we shall take the parameter ε to depend on n. The analysis shows that the appropriate scaling of ε is $n^{-1/3}$, so that it is natural to define yet another parameter $\lambda = \lambda_n$ according to

$$\varepsilon = \lambda_n n^{-1/3}, \tag{3}$$

and distinguish the cases λ_n bounded, $\lambda_n \to \infty$ and $\lambda_n \to -\infty$.

The main result is the following theorem.

Theorem 2.1 *There are constants ε_0 and λ_0, $0 < \varepsilon_0 < 1$, $0 < \lambda_0 < \infty$, such that*

$$\mathbb{P}(F_{n,m} \text{ is SAT}) = \begin{cases} 1 - \Theta\Big(\frac{1}{|\lambda_n|^3}\Big) & \text{if } -\varepsilon_0 n^{1/3} \le \lambda_n \le -\lambda_0 \\ \Theta(1) & \text{if } -\lambda_0 \le \lambda_n \le \lambda_0 \\ \exp\big(-\Theta\big(\lambda_n^3\big)\big) & \text{if } \lambda_0 \le \lambda_n \le \varepsilon_0 n^{1/3} \end{cases} . \tag{4}$$

Note that the behaviors for $\lambda_n < 0$ and $\lambda_n > 0$ can be cast in the same form by writing $\mathbb{P}(F_{n,m} \text{ is SAT}) = 1 - \Theta(|\lambda_n|^{-3}) = \exp(-\Theta(|\lambda_n|^{-3}))$.

Theorem 2.1 gives us the exact form of the scaling window:

Corollary 2.2 *For all sufficiently small $\delta > 0$, the scaling window (1) is of the form*

$$W(n,\delta) = (1 - \Theta(n^{-1/3}), 1 + \Theta(n^{-1/3})),$$

where the constants implicit in the definition of Θ depend on δ, and are easily calculated from equation (4).

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