A Family of Tests for IDMRL Alternatives with Unknown Turning Point Using Censored Data

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1. Introduction

We consider the problem of testing H_0 : F is the exponential distribution (i.e. $\bar{F}(x) = \exp(-x/\mu)$, $x \geq 0$, $\mu > 0$) against $H_1 : F$ is IDMRL, but not exponential based on a randomly censored data.

When complete data is utilized, Aly (1990), Hawkins, Kochar and Loader(henceforth HKL, 1992) and Na and Kim(2001) consider testing for exponentiality against IDMRL alternatives when neither the change point nor the proportion is known.

2. A Family of IDMRL Tests

Let $v(x) = \int_x^\infty \bar{F}(u)du$ and $f(x)$ denote the probability density function corresponding to F. As a measure of the deviation from H_0 in favor of H_1, we consider the parameter

$$T_j(F) = \sup\{\phi_j(p;F) : 0 \leq p \leq 1\}$$

where

$$\begin{aligned}\phi_j(p;F) &= \int_0^{F^{-1}(p)} \bar{F}^j(t)(f(t)v(t) - \bar{F}^2(t))dt + \int_{F^{-1}(p)}^\infty \bar{F}^j(t)(\bar{F}^2(t) - f(t)v(t))dt \\ &= \frac{1}{j+1}\Big(\int_0^\infty \bar{F}(t)dt - (j+2)\int_0^{F^{-1}(p)} \bar{F}^{j+2}(t)dt \\ &\quad +(j+2)\int_{F^{-1}(p)}^\infty \bar{F}^{j+2}(t)dt - 2\bar{F}^{j+1}(x)\int_{F^{-1}(p)}^\infty \bar{F}(t)dt\Big). \qquad (2.1)\end{aligned}$$

where j is a integer with $j > -1$. In our randomly censored model, we will replace F in (2.1) by the Kaplan-Meier estimator, $\hat{\bar{F}}_{KM}(x)$.

Now we propose the family of test statistics

$$T_j^c = \frac{\sup_{0\le p\le 1} \sqrt{n}\phi_j(p;\hat{F}_{KM})}{\hat{\mu}_F} \simeq \frac{\max_{0\le k\le n} \sqrt{n}(2\eta_j^c(k) - \eta_j^c(0))}{\hat{\mu}_F}$$

where $c_v = (n-v)/(n-v+1)$, $\hat{\mu}_F = \sum_{i=1}^n \{\prod_{v=1}^{i-1} {c_v}^{\delta_{(v)}}\}(Y_{(i)} - Y_{(i-1)})$, and

$$\eta_j^c(k) \;=\; \frac{1}{j+1}\sum_{i=k+1}^{n}\Big\{(j+2)\Big(\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big)^{j+2} - \Big(\prod_{v=1}^{k} c_v^{\delta_{(v)}}\Big)^{j+1}\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big\}(Y_{(i)} - Y_{(i-1)})$$

for $k = 0, 1, \cdots, n$.

Theorem 2.1 Under H_0, i.e. $F(x) = F_0(x) = \exp(-x/\mu)$,

$$T_j^c \xrightarrow{\mathcal{L}} Z_j^c \equiv \sup\{Z_j^c(p); 0 \le p \le 1\}$$

where $Z_j^c(p)$ denotes a mean zero Gaussian process with covariance

$$\begin{aligned}\sigma_j(p,q) \;&=\; \frac{1}{(j+1)^2}\Big\{\int_0^1 \frac{g_{1j}(z)}{\bar{H}(-\mu\log z)}dz + 4(1-\bar{p}^{j+1})\int_0^{\bar{q}} \frac{g_{2j}(q,z)}{\bar{H}(-\mu\log z)}dz \\ &+\; 2\int_0^{\bar{q}} \frac{g_{3z}g_{2j}(q,z)}{\bar{H}(-\mu\log z)}dz - 2\int_0^{\bar{p}} \frac{g_{3z}g_{2j}(p,z)}{\bar{H}(-\mu\log z)}dz\Big\},\end{aligned}$$

$g_{1j}(z) = (j+2)^2 z^{2j+3} - 2(j+2)z^{j+2} + z$, $g_{2j}(p,z) = (j+2)z^{j+2} - \bar{p}^{j+1}z$ and $g_{3j} = (j+2)z^{j+1} - 1$.

Using Durbin's (1985) approximation, we can obtain asymptotic critical values based on the distribution of T_j^c by

$$\Pr\{T_j^c > c\} \simeq \int_0^1 \frac{\exp\{-c^2/2\sigma_j(q,q)\}}{\sqrt{2\pi}\sqrt{\sigma_j(q,q)}}\Big\{c\Big(\frac{\partial}{\partial p}\sigma_j(p,q)\Big|_{p=q}\Big)\Big/\sigma_j(q,q)\Big\}dq.$$

Since the asymptotic critical values based on the distribution of T_j^c depends on H, we need a estimator of $\Pr\{T_j^c > c\}$. Using Rabinowitz's (1993) methodology of estimating Durbin's approximation, we can consistently estimate the asymptotic critical values based on the distribution of T_j^c by

$$\widehat{\Pr}\{T_j^c > c\} = \sum_{i=1}^n \frac{\exp\{-c^2/2\hat{\sigma}_j(p_i,p_i)\}}{\sqrt{2\pi}\sqrt{\hat{\sigma}_j(p_i,p_i)}}\frac{c}{\hat{\sigma}_j(p_i,p_i)}\Big(\hat{\sigma}_j(p_i,p_i) - \hat{\sigma}_j(p_{i-1},p_i)\Big)$$

where $p_i = i/n$.

Our large sample level α test for H_0 against H_1 is to reject H_0 in favor of H_1 if $\hat{p} \le \alpha$ where $\hat{p} = \widehat{\Pr}\{T_j^c > t_j^c\}$ and t_j^c is the observation of test statistic T_j^c.

Table 1 presents the empirical test size of IDMRL tests based on T_j^c for some different j with 25% censoring. The values in Tables 1 are the fraction of times that H_0 is rejected in favor of H_1 when H_0 is true. The empirical test sizes are calculated based on 1000 replications for; α =0.10, 0.05, 0.01; $n = 10, 20, \cdots, 100$.

Table 1. Empirical test size of IDMRL tests based on T_j^c.

n	α	j=0	j=1	j=2	j=3	j=4	j=5
	0.10	.198	.144	.131	.119	.118	.122
20	0.05	.113	.086	.071	.074	.070	.075
	0.01	.018	.027	.029	.031	.031	.033
	0.10	.183	.128	.117	.114	.122	.119
40	0.05	.090	.074	.066	.064	.063	.061
	0.01	.014	.015	.016	.019	.021	.021
	0.10	.159	.121	.106	.099	.098	.098
60	0.05	.071	.061	.051	.053	.049	.052
	0.01	.019	.009	.012	.016	.016	.017
	0.10	.170	.137	.126	.115	.105	.108
80	0.05	.084	.066	.062	.060	.060	.056
	0.01	.024	.015	.014	.013	.013	.015
	0.10	.180	.143	.113	.099	.096	.094
100	0.05	.102	.069	.055	.050	.047	.044
	0.01	.019	.016	.012	.012	.014	.015

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RESUME

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