Predictive HPD Matching Priors for Multivariate Elliptic Contoured Models

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1. Introduction

Probability matching priors, ensuring approximate frequentist validity of Bayesian credible regions have been of substantial current interest. Such priors are in a sense noninformative and can help in finding accurate frequentist confidence sets. Recently, Datta, Mukerjee, Ghosh and Sweeting (2000) addressed a related problem of characterizing matching priors when interest lies in predicting a future observation rather than estimating a parameter. In particular, in the multivariate case, these authors characterized, via an appropriate partial differential equation (PDE), priors ensuring frequentist validity, with margin of error $o(n^{-1})$, of predictive highest posterior density (HPD) regions for a future observation; here n is the number of past observations. These priors will be called predictive HPD matching priors.

In the present article we consider a multivariate elliptic contoured model specified by the density

$$f(x,q) = |\Sigma|^{-1/2}\, h[(x-m)'\, \Sigma^{-1}(x-m)], \quad x \in R^p, \qquad (1)$$

where $h(.)$ is a smooth, strictly decreasing, positive valued function defined over $[0,\infty)$, $\Sigma = ((s^{ij}))$ is a $p \times p$ positive definite (p.d.) matrix, and $m = (m_1,\dots,m_p)' \in R^p$. The parametric vector q consists of elements m_i $(1 \le i \le p)$ and s_{ij} $(1 \le i \le j \le p)$. For the model (1), we propose to characterize predictive HPD matching priors with margin of error $o(n^{-1})$. The handling of the PDE obtained in Datta et al (2000) gets rather complicated if one directly works with the parametrization in (1). We show how, with a convenient reparametrization, their PDE can be used in the present context. Note that predictive HPD matching priors are invariant under one-to-one transformation of q.

2. Main result

Since Σ^{-1} is p.d., it can be uniquely expressed as $\Sigma^{-1} = \Lambda'\Lambda$, where $\Lambda = ((l_{ij}))$ is a lower triangular matrix with diagonal elements all positive. Define

$$y_i = l_{ii}\ (1 \le i \le p),\ b_{ij} = l_{ij}/l_{ii}\ \ (1 \le j < i \le p), \qquad (2)$$

Let g be a parametric vector consisting of elements m_i, y_i $(1 \le i \le p)$ and b_{ij} $(1 \le j < i \le p)$. Then g is a one-to-one transformation of q. By (2), under the g-parametrization, the density (1) can be expressed as $\bar{f}(x,g) = (y_1 \dots y_p)\, h(y'y),\ x \in R^p$, where $y = (y_1,\dots,y_p)'$, with $y_1 = y_1(x_1 - m_1)$ and $y_i = y_i\{x_i - m_i + \sum_{j=1}^{i-1} b_{ij}(x_j - m_j)\}$ $(2 \le i \le p)$. Let $h_1(u) = \mathrm{d}\log h(u)/\mathrm{d}u$. Then

$$\partial \log \bar{f}(x,g)/\partial y_i = y_i^{-1}\{1 + 2y_i^2\, h_1(y'y)\}(1 \le i \le p),$$

$$\partial \log \bar{f}\ (x, g)/\partial m_p = -2 y_p y_p\ h_1(y'y),$$
$$\partial \log \bar{f}\ (x, g)/\partial m_i = -2(y_i\ y_i + \sum_{t=i+1}^{p} y_t\ b_{ti}\ y_t) h_1(y'y)\ (1 \le i \le p-1),$$
$$\partial \log \bar{f}\ (x, g)/\partial b_{ij} = y_i \{c_1(g)\ y_1 + \ldots + c_j(g) y_j\} h_1(y'y)\ (1 \le j < i \le p), \qquad (3)$$

where $c_1(g)$, $c_2(g), \ldots$, are parametric functions whose explicit forms will not be needed. From the above, one can check that the per observation Fisher information matrix for g takes the particularly simple form I = diag(I_1, I_2), where $I_1 = D^{-1}KD^{-1}$ corresponds to $y_i\ (1 \le i \le p)$ and I_2 corresponds to $m_i\ (1 \le i \le p)$ and $b_{ij}\ (1 \le j < i \le p)$. Here $D = \text{diag}(y_1, \ldots, y_p)$, K is a $p \times p$ p.d. matrix which does not involve g, the diagonal elements of K are all equal and the off-diagonal elements of K are also all equal. The explicit form of I_2 is not needed for our purpose.

For $0 < a < 1$, let $A(g, a) = \{x : \bar{f}\ (x, g) \ge m(g, a)\}$, where $m(g, a)$ is such that the integral of $\bar{f}\ (x, g)$ with respect to x over $A(g, a)$ equals a. Then, by (3), it can be seen that the integrals of $\partial \bar{f}(x, g)/\partial m_i\ (1 \le i \le p)$ and of $\partial \bar{f}(x, g)/\partial b_{ij}\ (1 \le j < i \le p)$ with respect to x over $A(g, a)$ vanish. Furthermore, for each $1 \le i \le p$, the integral of $\partial \bar{f}(x, g)/\partial y_i$ with respect to x over $A(g, a)$ equals $g(a)/y_i$, where $g(a)$ is a constant that depends on a but not on g. These facts, together with the form of the information matrix I as noted above, considerably simplify, in the present context, the PDE for predictive HPD matching priors as given in Datta et al (2000). Thus it can be seen that a prior $\bar{p}(g)$ is predictive HPD matching, with margin of error $o(n^{-1})$, if and only if

$$\sum_{i=1}^{p} \partial\{y_i\ \bar{p}(g)\}/\partial y_i = 0. \qquad (4)$$

We now return to the q-parametrization and consider the natural class of priors $p(q) = |\Sigma|^u$, where u is any real number. It can be seen that the unique prior in this class satisfying (4) under the g-parametrization corresponds to $u = -(p+1)/2$. Interestingly, this prior is in fact exact probability matching for predictive HPD regions under the multivariate elliptic contoured model. This exact result will be reported elsewhere.

REFERENCES

Datta G.S. Mukerjee, R., Ghosh, M. and Sweeting, T.J. (2000). Bayesian prediction with approximate frequentist validity. The Annals of Statistics 28, to appear.

RESUME

Pour les familles a symetrie elliptique, nous considerons les regions de plus haute densite a posteriori pour predire une future observation. Nous obtenons une loi a priori assurant la validite frequentiste de cette region.