INCREASING HAZARD FUNCTION RATIO AND INCREASING REVERSE HAZARD FUNCTION RATIO

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For a pair of hazard functions, or reverse hazard functions, the property of the title introduces orders between their distributions. The meaning of the orders is examines, and an application to the reliability of monitor is shown. For a p.d.f. $f(x)$ with the d.f. $F(x)$ and the s.f. $\overline{F}(x) = 1 - F(x)$ the hazard and the reverse hazard functions are denoted, respectively, by

$$h(x;\,\overline{F}) = f(x)/\overline{F}(x),\ \overline{F}(x) > 0, \quad \text{and} \quad \tilde{h}(x;\,F) = f(x)/F(x),\ F(x) > 0.$$

Definition 1

$$\overline{G} \succ \overline{F}\ \textit{`hf-ratio'} \quad \Leftrightarrow \quad h(x\,;\,\overline{G})/h(x\,;\,\overline{F}) = d\log\overline{G}/d\log\overline{F}\ \uparrow$$
$$G \succ F\ \textit{`rhf-ratio'} \quad \Leftrightarrow \quad \tilde{h}(x\,;\,G)/\tilde{h}(x\,;\,F) = d\log G/d\log F\ \uparrow$$

The order *'hf-ratio'* was proposed by Kalashnikov and Rachev (1986) and discussed by Sengupta and Deshpande (1994).

Proposition 1 Under the condition $\overline{G} \succ \overline{F}$ *'hf-ratio'* (or $G \succ F$ *'rhf-ratio'*) the following three cases U_h, V_h and W_h (or U_r, V_r and W_r) are possible:

If $\overline{G} \succ \overline{F}$ *'hf-ratio'*,	If $G \succ F$ *'rhf-ratio'*,
$U_h:\ \overline{G}(x)/\overline{F}(x)$ is nondecreasing,	$U_r:\ G(x)/F(x)$ is nondecreasing,
$V_h:\ \overline{G}(x)/\overline{F}(x)$ is nonincreasing,	$V_r:\ G(x)/F(x)$ is nonincreasing,
$W_h:\ \overline{G}(x)/\overline{F}(x)$ is unimodal.	$W_r:\ G(x)/F(x)$ is antiunimodal.

1. In the case U_h (or U_r) $h(x\,;\,\overline{G})/h(x\,;\,\overline{F})$ and $\overline{G}(x)/\overline{F}(x)$ (or $\tilde{h}(x\,;\,G)/\tilde{h}(x\,;\,F)$ and $G(x)/F(x)$) are nondecreasing, and the product $g(x)/f(x)$ is nondecreasing. Hence, G is larger than F in a strong sense.

2. In the case V_h (or V_r) $\overline{G}(x) \leq \overline{F}(x)$, $\forall x$, and g/f cannot be increasing.
3. In the case W_h, $\overline{G}(x) \geq \overline{F}(x)$ for smaller x, but eventually $\overline{G}(x) \leq \overline{F}(x)$ and the 'life' of $\overline{G}$ is shorter than that of $\overline{F}$. This fact was noted by Kalashnikov and Rachev (1986) in relation with life data analysis, since the survival functions cross sometime later. In this case and in the case V_h, $\overline{G}$ is smaller than $\overline{F}$ and contradicts to our notation emphasizing the case U_h.
4. In the case W_r, conversely, $G(x) \geq F(x)$ for smaller x, and eventually $G(x) \leq F(x)$. Hence $\overline{G}$ is finally larger than $\overline{F}$.

Let $(Y_1, \ldots, Y_n)$ be random sample from a continuous d.f. K which is F or G. Let $(\xi_1, \ldots, \xi_n)$ be a sequence of threshold values such that $0 < F(\xi_j) < 1$ and $0 < G(\xi_j) < 1$, $j = 1, \ldots, n$. In testing a statistical problem

$$H_0 \quad : \quad K = F \quad \text{vs} \quad K = G, \quad F \text{ and } G \text{ are known},$$

only the binary data $T_j = I[Y_j > \xi_j]$, $j = 1, \ldots, n$, are observed. $I[\cdot]$ is a predicate with the value 1 or 0, if the condition in $[\cdot]$ is true or false, respectively.

Proposition 2 (Sibuya and Suzuki, 2001)
1. Suppose that $\overline{G} \succ \overline{F}$ *'hf-ratio'*, and H_0 is rejected if $\sum_{j=1}^{n} T_j = n$. Among the test of level α, $\xi_1 = \cdots = \xi_n = a$, $1 - F(a) = \alpha^{1/n}$ maximizes the power of the test.
2. Suppose that $G \succ F$ *'rhf-ratio'* and H_0 is rejected if $\sum_{j=1}^{n} T_j \geq 1$. Among the test of level α, $\xi_1 = \cdots = \xi_n = a$, $F(a) = F(1-\alpha)^{1/n}$ maximizes the power.

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