Asymptotics of the Overflow Probability in Load-Balanced Jackson Networks

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1. Introduction

We analyze a rare-event probability in queueing networks. The probability in question is

$$\begin{aligned} p_K \;&:=\; P\{\text{network population reaches } K \text{ before returning to } 0, \\ &\quad\;\; \text{starting from } 0\}, \end{aligned}$$

a type of *overflow probability* if we think of K as an upper limit on the network population. The network we consider is a Jackson network of n nodes or service stations that operate on a first-come-first-served basis. Customers arrive at a typical node i from outside the system according to a Poisson process with rate $\bar{\lambda}_i$ and, if necessary, wait in a buffer to get served until the station gets free. Service time is exponentially distributed with mean $1/\mu_i$. Once service is completed, the customer is routed to another node, say j, with probability r_{ij} or leaves the system with probability $r_{i.} := 1 - \sum_{j=1}^{n} r_{ij}$. The Jackson network can be described as a Markov jump process $\{X(t); t \geq 0\}$ on $\mathcal{S} \equiv \mathbf{N}^n$, where the state $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{S}$ depicts the system when there are x_i customers waiting or being served at node i.

Jackson(1957) gave an expression for the invariant measure of a Jackson network. For an exogenously supplied and open Jackson network for which the solution $(\lambda_1, \cdots, \lambda_n)$ to the traffic equations

$$\lambda_i = \bar{\lambda}_i + \sum_{j=1}^{n} \lambda_j r_{ji}, \quad i = 1, 2, \cdots, n$$

satisfies the light traffic conditions

$$\rho_i := \frac{\lambda_i}{\mu_i} < 1, \quad i = 1, 2, \cdots, n, \tag{1}$$

the stationary distribution $\pi(\vec{x})$ of $\vec{x} = (x_1, \cdots, x_n) \in \mathcal{S}$ is given by the product

$$\pi(\vec{x}) = \prod_{i=1}^{n} (1 - \rho_i)\rho_i^{x_i}.$$

The ratio ρ_i is called the *load* on node i. We assume the light traffic conditions (1) hold. We further assume that s nodes, say $\{1, 2, \cdots, s\}$, have the same maximal load ρ_*, that is, $\rho_* = \rho_1 = \rho_2 = \cdots = \rho_s > \rho_{s+1} \geq \cdots \geq \rho_n$.

In section 2, we derive upper and lower bounds on p_K in two steps; we first bound the stationary probability of the set of states with population K. We, then, use the time reversal and the fluid limit in Anantharam et al.(1990) to convert the stationary bounds to the bounds on the transient probability p_K, which imply the stronger logarithmic limit for p_K than that obtained by Glasserman and Kou(1995).

2. Asymptotics of the overflow probability

Define the *overflow set*

$$C_K = \{\vec{x} \in \mathcal{S} : x_1 + x_2 + \cdots + x_n = K\},$$

the set of states in which the network population is exactly K.

Lemma

$$b_1 \rho_*^K (K+1)^{s-1} \leq \pi(C_K) \leq b_2 \rho_*^K (K+1)^{s-1}$$

for positive constants b_1 *and* b_2.

Theorem *Consider an exogenously supplied and open Jackson network which satisfies the light traffic conditions. Then, for some positive constants* c_1, c_2 *that do not depend on the population size* K *but may depend on the network parameters, we have*

$$c_1 \pi(C_K) \leq p_K \leq c_2 \pi(C_K).$$

Combining Lemma and Theorem gives the following corollary.

Corollary *For an exogenously supplied and open stable Jackson network in which* s *nodes have the same maximal load* ρ_*,

$$\lim_{K \to \infty} \frac{\log p_K - \log \rho_*^K}{\log K} = s - 1.$$

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