Asymptotics of the Overflow Probability in Load-Balanced Jackson Networks

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1. Introduction

We analyze a rare-event probability in queueing networks. The probability in question is

\[ p_K := P\{\text{network population reaches } K \text{ before returning to } 0, \text{ starting from } 0\}, \]

a type of overflow probability if we think of \( K \) as an upper limit on the network population. The network we consider is a Jackson network of \( n \) nodes or service stations that operate on a first-come-first-served basis. Customers arrive at a typical node \( i \) from outside the system according to a Poisson process with rate \( \bar{\lambda}_i \) and, if necessary, wait in a buffer to get served until the station gets free. Service time is exponentially distributed with mean \( 1/\mu_i \). Once service is completed, the customer is routed to another node, say \( j \), with probability \( r_{ij} \) or leaves the system with probability \( r_i := 1 - \sum_{j=1}^{n} r_{ij} \). The Jackson network can be described as a Markov jump process \( \{X(t); t \geq 0\} \) on \( S = \mathbb{N}^n \), where the state \( \vec{x} = (x_1, x_2, \ldots, x_n) \in S \) depicts the system when there are \( x_i \) customers waiting or being served at node \( i \).

Jackson(1957) gave an expression for the invariant measure of a Jackson network. For an exogenously supplied and open Jackson network for which the solution \( (\lambda_1, \ldots, \lambda_n) \) to the traffic equations

\[ \lambda_i = \bar{\lambda}_i + \sum_{j=1}^{n} \lambda_j r_{ji}, \quad i = 1, 2, \ldots, n \]

satisfies the light traffic conditions

\[ \rho_i := \frac{\lambda_i}{\mu_i} < 1, \quad i = 1, 2, \ldots, n, \quad (1) \]

the stationary distribution \( \pi(\vec{x}) \) of \( \vec{x} = (x_1, \ldots, x_n) \in S \) is given by the product

\[ \pi(\vec{x}) = \prod_{i=1}^{n} (1 - \rho_i)^{r_{i}}. \]
The ratio $\rho_i$ is called the load on node $i$. We assume the light traffic conditions (1) hold. We further assume that $s$ nodes, say $\{1, 2, \cdots, s\}$, have the same maximal load $\rho_s$, that is, $\rho_s = \rho_1 = \rho_2 = \cdots = \rho_s > \rho_{s+1} \geq \cdots \geq \rho_n$.

In section 2, we derive upper and lower bounds on $p_K$ in two steps; we first bound the stationary probability of the set of states with population $K$. We then, use the time reversal and the fluid limit in Anantharam et al. (1990) to convert the stationary bounds to the bounds on the transient probability $p_K$, which imply the stronger logarithmic limit for $p_K$ than that obtained by Glasserman and Kou (1995).

2. Asymptotics of the overflow probability

Define the overflow set

$$C_K = \{\vec{x} \in \mathcal{S} : x_1 + x_2 + \cdots + x_n = K\},$$

the set of states in which the network population is exactly $K$.

Lemma

$$b_1 \rho^K_s (K+1)^{s-1} \leq \pi(C_K) \leq b_2 \rho^K_s (K+1)^{s-1}$$

for positive constants $b_1$ and $b_2$.

Theorem Consider an exogenously supplied and open Jackson network which satisfies the light traffic conditions. Then, for some positive constants $c_1, c_2$ that do not depend on the population size $K$ but may depend on the network parameters, we have

$$c_1 \pi(C_K) \leq p_K \leq c_2 \pi(C_K).$$

Combining Lemma and Theorem gives the following corollary.

Corollary For an exogenously supplied and open stable Jackson network in which $s$ nodes have the same maximal load $\rho_s$,

$$\lim_{K \to \infty} \frac{\log p_K - \log \rho^K_s}{\log K} = s - 1.$$ 

REFERENCES

