Split-Plot Design under Nonnormality

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1.Introduction: In design of experiment studies, most of the literature based on the normality assumption of the error distribution. Hovewer, in the real life applications nonnormal errors are most commonly observed. In this paper a specific form of design of experiments, split-plot design, is considered with nonnormal errors assumption. The distributional approach for the errors leads the authors to work with Weibull with support IR: $(0,\infty)$ which is a skewed distribution.

This study proposes an alternative method to estimate the parameters by using Modified Maximum Likelihood (MML) technique when maximum likelihood (ML) technique does not work due to its likelihood equations are intractable and solving ML estimators by iterative methods can be problematic. MML (Tiku, 1967, 1968) firstly expresses the likelihood equations in terms of order statistics, then linearizes the intractable functions by using the first two terms of a Taylor series expansion and finally incorporates these terms to likelihood equations. For estimating the location and scale parameters of location-scale distributions, the modified likelihood equations have explicit solutions. MML estimators are easy to compute and known to be asymptotically fully efficient under regularity conditions (Bhattacharyya, 1985; Vaughan, 1992) and almost as efficient as the ML estimators for small sample sizes (Tiku et al., 1986).

2.The Model: Nonnormal error assumption and application of MML technique in two-way classification with interaction has recently been done by Þenoðlu (2000). Here, the methodology is extended to split-plot design problems. The linear model for the split-plot design is as follows:

$$y_{ijk} = \mu + r_i + \alpha_j + (r\alpha)_{ij} + \beta_k + (\alpha\beta)_{jk} + e_{ijk}\ ; \qquad i=1,..,r;\ j=1,..a;\ k=1,..,b \tag{1}$$

where $(r\alpha)_{ij}$’ s (main plot errors) and $e_{ijk's}$ (subplot errors) are i.i.d. and their distributions are Weibull, $W(p,\sigma_A)$ and $W(p,\sigma_B)$, respectively. The family of Weibull distribution is given by,

$$\frac{p}{\sigma^p} e^{p-1} \exp\left\{-\left(\frac{e}{\sigma}\right)^p\right\},\ 0 < e < \infty\ \ (p > 0). \tag{2}$$

Since the complete sums are invariant to ordering, the likelihood equations are

$$\frac{\partial \ln L}{\partial r_i} = -\frac{p-1}{\sigma_B}\sum_{j=1}^{a}\sum_{k=1}^{b} z_{ij(k)}^{-1} + \frac{p}{\sigma_B}\sum_{j=1}^{a}\sum_{k=1}^{b} z_{ij(k)}^{p-1} = 0\,; \frac{\partial \ln L}{\partial \alpha_j} = -\frac{p-1}{\sigma_B}\sum_{i=1}^{r}\sum_{k=1}^{b} z_{ij(k)}^{-1} + \frac{p}{\sigma_B}\sum_{i=1}^{r}\sum_{k=1}^{b} z_{ij(k)}^{p-1} = 0$$

$$\frac{\partial \ln L}{\partial (r\alpha)_{ij}} = -\frac{p-1}{\sigma_B}\sum_{k=1}^{b} z_{ij(k)}^{-1} + \frac{p}{\sigma_B}\sum_{k=1}^{b} z_{ij(k)}^{p-1} = 0\,; \frac{\partial \ln L}{\partial \beta_k} = -\frac{p-1}{\sigma_B}\sum_{i=1}^{r}\sum_{j=1}^{a} z_{ij(k)}^{-1} + \frac{p}{\sigma_B}\sum_{i=1}^{r}\sum_{j=1}^{a} z_{ij(k)}^{p-1} = 0\,;$$

$$\frac{\partial \ln L}{\partial (\alpha\beta)_{jk}} = -\frac{p-1}{\sigma_B}\sum_{i=1}^{r} z_{ij(k)}^{-1} + \frac{p}{\sigma_B}\sum_{i=1}^{r} z_{ij(k)}^{p-1} = 0\,. \tag{3}$$

where $z_{ij(k)} = (y_{ij(k)} - \mu - r_i - \alpha_j - (r\alpha)_{ij} - \beta_k - (\alpha\beta)_{jk})/\sigma_B$, $(1 \le k \le b)$. To derive modified likelihood equations, we linearize the function $z_{ij(k)}^{-1}$ and $z_{ij(k)}^{p-1}$ as follows:

$z_{ij(k)}^{p-1} \cong t_{(k)}^{p-1} + \left[z_{ij(k)} - t_{(k)}\right]\left\{\frac{d}{dz} z_{ij(k)}^{p-1}\right\}_{z_{ij(k)} = t_{(k)}} = \alpha_k + \beta_k z_{ij(k)}$. Here, $t_{(k)} = E\{z_{ij(k)}\}$ is the expected value of the ith order statistic $z_{ij(k)}$ and $\alpha_k = (2-p)t_{(k)}^{p-1}$ and $\beta_k = (p-1)t_{(k)}^{p-2}$, $1 \le k \le b$.

Similarly, $z_{ij(k)}^{-1} \cong \alpha_{k0} - \beta_{k0} z_{ij(k)}$. Here, $\alpha_{k0} = 2t_{(k)}^{-1}$, $\beta_{k0} = t_{(k)}^{-2}$ and $t_{(k)} = \left[-\ln(1-k/n+1)\right]^{1/p}$.

Incorporating the linear approximations in the likelihood equations above, the modified likelihood equations are obtained. The solutions of these equations are the MML estimators:

$$\hat\mu = \hat\mu_{...} - \frac{\Delta}{m}\hat\sigma_B\ ;\quad \hat r_i = \hat\mu_{i..} - \hat\mu_{...}\ ;\quad \hat\alpha_j = \hat\mu_{.j.} - \hat\mu_{...};\quad (\widehat{r\alpha})_{ij} = \hat\mu_{ij.} - \hat\mu_{i..} - \hat\mu_{.j.} + \hat\mu_{...};$$

$$\hat\beta_k = \hat\mu_{..k} - \hat\mu_{...};\quad (\widehat{\alpha\beta})_{jk} = \hat\mu_{.jk} - \hat\mu_{.j.} - \hat\mu_{..k} + \hat\mu_{...}\ ;\quad \hat\sigma_B = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N - ra - ab + a)}} \tag{4}$$

where $\Delta = \sum_{k=1}^{b}\Delta_k$, $\Delta_k = (p-1)\alpha_{k0} - p\alpha_k$, $m = \sum_{k=1}^{b}\delta_k$, $\delta_k = (p-1)\beta_{k0} + p\beta_k$ and the divisor N in $\hat\sigma_B$ is replaced by $\sqrt{N(N - ra - ab + a)}$ as a bias correction. $\hat\mu_{...} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{a}\sum_{k=1}^{b}\delta_k y_{ij(k)}}{ram}$,

$$\hat\mu_{i..} = \frac{\sum_{j=1}^{a}\sum_{k=1}^{b}\delta_k y_{ij(k)}}{am},\ \hat\mu_{.j.} = \frac{\sum_{i=1}^{r}\sum_{k=1}^{b}\delta_k y_{ij(k)}}{rm},\ \hat\mu_{ij.} = \frac{\sum_{k=1}^{b}\delta_k y_{ij(k)}}{m},\ \hat\mu_{.jk} = \frac{\sum_{i=1}^{r}\delta_k y_{ij(k)}}{r}, \tag{5}$$

$$A=N,\quad B = \sum_{i=1}^{r}\sum_{j=1}^{a}\sum_{k=1}^{b}(y_{ij(k)} - \hat\mu_{ij.} - \hat\mu_{.jk} + \hat\mu_{.j.})\Delta_k,\quad C = \sum_{i=1}^{r}\sum_{j=1}^{a}\sum_{k=1}^{b}(y_{ij(k)} - \hat\mu_{ij.} - \hat\mu_{.jk} + \hat\mu_{.j.})^2 \boldsymbol{d}_k$$

From the results given above, $\hat\sigma_A^2 = \frac{1}{m}\left[\frac{m\sum_{i=1}^{r}\sum_{j=1}^{m}(\hat\mu_{ij.} - \hat\mu_{i..} - \hat\mu_{.j.} + \hat\mu_{...})^2}{(r-1)(a-1)\left[\Gamma\left(1+\frac{2}{p}\right) - \Gamma^2\left(1+\frac{1}{p}\right)\right]} - \hat\sigma_B^2\right]$ is obtained.

REFERENCES

Bhattacharyya, G.K. (1985). The asymptotics of maximum likelihood and related estimators based on Type II censored data. J.Amer. Stat. Assoc., 80, 398-404.

Þenoðlu, B. (2000). Experimental Design Under Nonnormality, PhD. Thesis, METU, Turkey.

Vaughan, D.C.(1992). On the Tiku-Suresh method of estimation.Commun.Stat.-Theory Meth. 21,451-469.

Tiku, M.L. (1967). Estimating the mean and standard deviation from censored normal samples. Biometrika 54, 155-165.

Tiku, M.L. (1968). Estimating the parameters of log-normal distribution from censored samples. J. Amer. Stat Assoc. 63, 134-140.

Tiku, M.L., Tan, W.Y., and Balakrishnan, N. (1986). Robust Inference. New York: Marcel Dekker.

RESUME: *Cette étude introduit une méthode efficace pour evaluer une estimation des paramètres pour les modèles des parcelles divisibles, en utilisant la technique de maximum vraisemblance modifiée MML, (Tiku 1967) pour non-normales situations. Les estimateurs MML, sont des fonctions explicites des observations d'échantillons, sont alors faciles à obtenir.*