1 Moment-Inequalities of A Random Variable Defined Over A Finite Interval

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1. INTRODUCTION

Set X to denote a random variable whose probability function is $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ and its associated distribution function $F : [0, 1]$. Denote by M_r the r^{th} central moment of the random variable X defined as $M_r = \int_a^b (t - \mu)^r dF$; $r = 0, 1, 2, \ldots$, where μ is the mean of the random variable X. We present estimations for M_r and some inequalities involving M_r.

2. RESULTS INVOLVING HIGHER MOMENTS

An identity involving higher central moments:

Theorem 1 2.1. For the random variable X with the assumptions as stated above, we have

$$\int_a^b (b - t)(t - a)^m dF = \sum_{k=0}^{m} \binom{m}{k} (\mu - a)^k [(b - \mu) M_{m-k} - M_{m-k+1}]; \quad m = 1, 2, 3, \ldots \quad (2.1)$$

where M_{m-k} and M_{m-k+1} are respectively the $(m - k)^{th}$ and $(m - k + 1)^{th}$ central moments of the random variable X.

3. SOME ESTIMATIONS FOR THE CENTRAL MOMENTS

$$0 \le M_2 \le (b - \mu)(\mu - a) \le \frac{(b - a)^2}{4}. \quad (3.1)$$

$$M_3 \le (b - \mu)(\mu - a)(a + b - 2\mu). \quad (3.2)$$

$$M_3 \le \frac{1}{4}[(b - \mu)^3 + (b - \mu)(\mu - a)^2 - 2(\mu - a)^3]. \quad (3.3)$$

$$M_4 \le (b - \mu)(\mu - a)[(b - a)^2 - 3(b - \mu)(\mu - a)]. \quad (3.4)$$

$$M_4 \le \frac{1}{4}[(b - \mu)^4 + 4(b - \mu)^2(\mu - a)^2 - 4(b - \mu)(\mu - a)^3 + 3(\mu - a)^4]. \quad (3.5)$$

The same bounds as (3.1) for M_2 were also obtained by Barnett and Dragomir [1].

4. RESULTS BASED ON THE GRÜSS & HÖLDER'S INEQUALITIES

Lemma 2 4.1. For the random variable X with the above assumptions, we have

$$\int_a^b (b - t)^r (t - a)^s dt = (b - a)^{r+s+1} \cdot \frac{\Gamma(r + 1)\Gamma(s + 1)}{\Gamma(r + s + 2)}; \quad r, s \ge 0; \quad (4.1)$$

Theorem 3 4.2. For the random variable X with the above assumptions,

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\cdot\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right|$$
$$\le \frac{1}{2}(M-m)(b-a)^{r+s+1}\cdot\left[\frac{\Gamma(2r+1)\Gamma(2s+1)}{\Gamma(2r+2s+2)} - \left(\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right)^2\right]^{1/2}; \quad r,s\ge 0; \quad (4.2)$$

where $m \le f \le M$ a.e. on $[a,b]$.

Considering $r = s = 1$ in (4.2) provides Theorem 1 by Barnett and Dragomir [1].

Lemma 4 4.3. For the random variable X with the above assumptions, holds the inequality

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\cdot\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right|$$
$$\le \frac{1}{2}(M-m)\left[(b-a)\int_a^b f^2(t)\,dt - 1\right]^{1/2}; \quad r,s\ge 0; \quad (4.3)$$

where $m \le f \le M$ a.e. on $[a,b]$.

Theorem 5 4.4. For the random variable X with the above assumptions,

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\cdot\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right|$$
$$\le \frac{1}{4}(b-a)(M-m)^2; \quad r,s\ge 0. \quad (4.4)$$

Theorem 6 4.5. For the random variable X with the above assumptions,

$$\left|\int_a^b (b-t)^r (t-a)^s f(t)dt - (b-a)^{r+s}\cdot\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\right|$$
$$\le \frac{1}{4}M(M-m)(b-a); \quad r,s\ge 0. \quad (4.5)$$

Theorem 7 4.6. For the random variable X with the above assumptions, suppose that f is $n-$ times differentiable and $f^{(n)}$ $(n \ge 0)$ is absolutely continuous on $[a,b]$. Then,

$$\left|\int_a^b (t-a)^r (b-t)^s f(t)dt - \sum_{k=0}^{n} (b-a)^{r+s+k+1}\cdot\frac{\Gamma(s+1)\Gamma(r+k+1)}{\Gamma(r+s+k+2)}\right|$$
$$\le \frac{1}{n!}\cdot\begin{cases} \frac{\|f^{(n+1)}\|_\infty}{n+1}\cdot(b-a)^{r+s+n+2}\cdot\frac{\Gamma(r+n+2)\Gamma(s+1)}{\Gamma(r+s+n+3)}; & \text{if } f^{(n+1)}\in L_\infty[a,b];\\ \frac{\|f^{(n+1)}\|_p}{(nq+1)^{1/q}}\cdot(b-a)^{r+s+n+\frac{1}{q}+1}\cdot\frac{\Gamma(r+n+\frac{1}{q}+1)\Gamma(s+1)}{\Gamma(r+s+n+\frac{1}{q}+2)}; & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1;\\ \|f^{(n+1)}\|_1\cdot(b-a)^{r+s+n+1}\cdot\frac{\Gamma(r+n+1)\Gamma(s+1)}{\Gamma(r+s+n+2)}; & \text{if } f^{(n+1)}\in L_1[a,b]; \end{cases} \quad (4.6)$$

where $\|\cdot\|_p$ $(1 \le p \le \infty)$ are the Lebesgue norms on $[a,b]$, i.e.,

$$\|g\|_\infty := \operatorname{ess}\sup_{t\in[a,b]} |g(t)|; \text{ and } \|g\|_p := \left(\int_a^b |g(t)|^p dt\right)^{1/p}; \quad (p \ge 1).$$

Considering $r = s = 1$, (4.6) leads to the Theorem 3 of Barnett and Dragomir [1].

REFERENCES

[1] Barnett, N.S. and Dragomir, S.S. (1999). Some elementary inequalities for the expectation and variance of a random variable whose pdf is defined on a finite interval. RGMIA Res. Rep. Coll.,2(7).[ONLINE] http://rgmia.vu.edu.au/v1n2.html

[2] Barnett, N.S., Cerone, P., Dragomir, S.S. and Roumeliotis, J. (2001). Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval. J. Ineq. Pure & Appl. Math., 2(1),1-18.[ONLINE] http://rgmia.vu.edu.au/v1n2.html

RESUME

Some estimations for the higher order central moments of a random variable when the random variable is defined over a finite interval are given. Further, some additional inequalities involving these moments are established.