On finite difference approximations of stochastic PDEs

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1. Introduction

Stochastic methods have become increasingly important in the analysis of a broad range of phenomena in natural sciences and economics. Many systems are described by differential equations where some of the parameters and/or the initial data are not known with complete certainty due to the lack of information, uncertainty in the measurements or incomplete knowledge of the mechanisms themselves. To compensate for this lack of information one introduces stochastic noise in the equations. This results in *stochastic differential equations*.

Since Itô's fundamental work on finite dimensional stochastic calculus around 1940s, stochastic *ordinary* differential equations and their applications have been extensively studied and there exist huge literature devoted to this subject. We refer the reader to books by Karatzas-Shreve [K-S], Øksendal [Oks], Kloeden-Platen [K-P] and references therein.

The necessity of considering equations combining the features of *partial differential equations* (PDEs) and Itô equations arose both in the theory of stochastic processes and in natural sciences and engineerings in mid 60's. Such equations appeared in filtering problem, statistical hydrodynamics, population genetics, Euclidean field theory, control theory and other fields. These equations describe the evolution in time of systems with values in some function spaces (*evolutional stochastic partial differential equations*, SPDEs). Basic theoretical questions on existence, uniqueness and regularity of solutions have been asked and answered under various set of conditions in the 1970s and 1980s and are still of great interest today. The reader is referred to books by Da Prato-Zabczyk [D-Z], Rozovskii [Roz].

This paper concerns the finite difference approximations of stochastic partial differential equations. By replacing the space derivatives and the stochastic differentials with suitable difference quotients, one obtains *discrete* stochastic equations or stochastic partial *difference* equations. The finite difference scheme is one of the most frequently used methods for finite dimensional approximations of (deterministic) elliptic and parabolic PDEs. See Richtmyer-Morton [R-M]. Many finite difference schemes currently in use for the numerical solutions of (stochastic) partial differential equations with constant coefficients can be successfully analyzed for stability by means of Fourier methods. But it is in general a non-trivial task to rigorously verify the stability and convergence for equations with variable coefficients. The techniques for obtaining a priori integral estimates (L^2-theory) for solutions of (stochastic) parabolic differential equations do not require the equations to have constant coefficients. Therefore it is reasonable to expect that if these techniques can be carried over to the finite difference equations, they might provide an effective way for studying the stability and convergence properties of finite difference schemes for equations with variable coefficients.

It is the purpose of this paper to show that such an analogy indeed exists for stochastic PDEs. Employing developed L^2-theory of discrete equations, we obtain error estimates and a rate of convergence.

Let $(\Omega,\, \mathcal{F},\, P)$ be a complete probability space, $(\{\mathcal{F}_t\}, t \geqslant 0)$ be an increasing filtration of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω. Let $\{w_t^k; k = 1, 2, \ldots, d'\}$ be independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega,\, \mathcal{F},\, P)$. For the above standard terminologies, the reader is referred to Karatzas-Shreve [K-S].

2. Stochastic partial difference equations and L^2-theory

We introduce some operators acting on functions on $\mathbb{Z}_h^d := \{hm = h(m_1, m_2, \ldots, m_d) : m \in \mathbb{Z}^d\}$ for some positive constant $h \in (0,1)$. Let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^d$. For $i \in \{1, \ldots, d\}$ and a real-valued function v on $\mathbb{Z}_h^d$, the finite difference operators are defined by

$$\begin{cases} \delta_h^{+,e_i} v(x) := \delta_{h,i}^{+} v(x) := \frac{1}{h}[v(x + he_i) - v(x)] \\ \delta_h^{-,e_i} v(x) := \delta_{h,i}^{-} v(x) := \frac{1}{h}[v(x) - v(x - he_i)]. \end{cases}$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we define $\delta^\alpha := \delta_1^{\alpha_1} \circ \delta_2^{\alpha_2} \circ \ldots \circ \delta_d^{\alpha_d}$, where $\delta_i^{\alpha_i} := \delta_i^{+} \circ \ldots \circ \delta_i^{+}$. For a nonnegative integer n, the space $H^n_{dis,h}$ is defined as the set of all functions v such that

$$\| v \|^2_{H^n_{dis,h}} := \sum_{|\alpha| \leqslant n} \| \delta^\alpha v \|^2_{L_2} < \infty.$$

The spaces $H^n_{dis,h}$ are Hilbert spaces with norms $\| \cdot \|_{H^n_{dis,h}}$.

We consider SPDEs of the form

$$du = \Big(\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i} \Big)\, dt + \sum_{k=1}^{d'} \big(\sum_{i=1}^{d} (\sigma_i^k \frac{\partial u}{\partial x_i}) + g^k \big)\, dw_t^k,$$

$$u(0, \cdot) = u_0$$

and their finite difference approximations

$$\begin{aligned} & u_h^{\Delta t}(t_{l+1}, x_m) - u_h^{\Delta t}(t_l, x_m) \\ &= \sum_{i=1}^{d} \big[\delta_{h,i}^{-} \big(\sum_{j=1}^{d} a_{ij}(t_l, x_m)\, \delta_{h,j}^{+}\, u_h^{\Delta t}(t_l, x_m) \big) + \delta_{h,i}^{-}\, f_i(t_l, x_m) \big]\, \Delta t \\ &+ \sum_{k=1}^{d'} \big[\sum_{i=1}^{d} \sigma_i^k(t_l, x_m)\, \delta_{h,i}^{+}\, u_h^{\Delta t}(t_l, x_m) + g^k(t_l, x_m) \big]\, \Delta w_l^k. \end{aligned}$$

where $\Delta t := t_{l+1} - t_n$, $t_l := \frac{T}{N} l$, $l = 0, 1, 2, \ldots, N$, $\Delta w_l^k := w_{t_{l+1}}^k - w_{t_l}^k$ and $a_{ij}, \sigma_i^k, f_i, g^k$ are functions defined on $\Omega \times [0, T] \times \mathbb{R}^d$. We assume that there exist positive constants λ and Λ such that for any $\xi \in \mathbb{R}^d$ and ω, t, x,

$$\lambda |\xi|^2 \leqslant (2a_{ij} - \alpha_{ij}) \xi_i \xi_j \leqslant \Lambda |\xi|^2,$$

where $\alpha_{ij} := \sigma_i^k \sigma_j^k$. Note that we used the summation convention. Let n be a nonnegative integer. We assume that for any $|\beta| \leqslant n$, $\delta^\beta\, a_{ij}$ and $\delta^\beta\, \alpha_{ij}$ are bounded. Let

$$K_n := \sup_{|\beta| \leqslant n} \sup_{\omega, t, x} \{ |\delta^\beta\, a_{ij}|, \, |\delta^\beta\, \alpha_{ij}| \} < \infty.$$

We also assume that $u^0 \in H^n_{\mathrm{dis},h}$, $f_i(t, \cdot) \in H^n_{\mathrm{dis},h}$, $f(t, \cdot) \in H^n_{\mathrm{dis},h}$, $g^k(t, \cdot) \in H^n_{\mathrm{dis},h}$ and moreover

$$E \int_0^T \sum_{i=1}^d \| \, f_i(t, \cdot) \, \|^2_{H^n_{\mathrm{dis},h}} + \| \, f(t, \cdot) \, \|^2_{H^n_{\mathrm{dis},h}} + \sum_{k=1}^{d'} \| \, g^k(t, \cdot) \, \|^2_{H^n_{\mathrm{dis},h}} \; dt < \infty.$$

Let $\theta := \Delta t / h^2$. Suppose that θ is "sufficiently small".

Theorem 1.

For a solution $u^n := u(t_n, \cdot)$ of the above finite difference equation, we have

$$E \sup_{0 \leqslant n \leqslant N} \| \, u^n \, \|^2_{H^n_{\mathrm{dis},h}} + E \sum_{n=0}^N \| \, u^n \, \|^2_{H^{n+1}_{\mathrm{dis},h}} \; \Delta t \leqslant N \big(E \, \| \, u^0 \, \|^2_{H^n_{\mathrm{dis},h}}$$

$$+ E \int_0^T \sum_{i=1}^d \| \, f_i(t) \, \|^2_{H^n_{\mathrm{dis},h}} \, dt + E \sum_{n=0}^N \| \, f^n \, \|^2_{H^n_{\mathrm{dis},h}} \; \Delta t + E \int_0^T \sum_{k=1}^{d'} \| \, g^k(t) \, \|^2_{H^n_{\mathrm{dis},h}} \; dt \big),$$

where $N = N(\lambda, K_n, d, T)$.

3. Error estimates

Employing Theorem 1, the L^2-theory of stochastic PDEs (see Rozovskii [Roz]) and some embedding relations for discrete function spaces (see Stevenson [Ste]) we obtain the following error estimate.

Theorem 2.

Suppose that $n > k + d + 2$. Then for any multi-index α such that $|\alpha| \leqslant k$, the error $u(t_l, x_m) - u_h^{\Delta t}(t_l, x_m)$ satisfies

$$E \sup_{l \leqslant N} \sup_{m \in \mathbb{Z}^d} | \, \delta^\alpha \big(u(t_l, x_m) - u_h^{\Delta t}(t_l, x_m) \big) \, |^2$$

$$\leqslant N h^2 \big(E \, \| \, u_0 \, \|^2_{n,2} + E \int_0^T \sum_{i=1}^d (\| \, f_i(t, \cdot) \, \|^2_{n,2} + \| \, \frac{\partial f_i}{\partial t}(t, \cdot) \, \|^2_{n,2}) \, dt$$

$$+ E \int_0^T \sum_{k=1}^{d'} (\| \, g^k(t, \cdot) \, \|^2_{n,2} + \| \, \frac{\partial g^k}{\partial t}(t, \cdot) \, \|^2_{n,2}) \, dt \big),$$

where $N = N(\lambda, d, K_n, T, n, k)$.

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RESUME

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