NONPARAMETRIC ESTIMATION OF COPULAS FOR TIME SERIES

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1. Outline

Knowledge of the dependence structure between financial assets or claims is crucial to achieve performant risk management in finance and insurance. To measure dependence through computation of correlations reveals adequate in the context of multivariate normally distributed risks or in assessing linear dependence. Contemporary financial risk management however calls for other tools due to the presence of an increasing proportion of nonlinear risks (derivative assets) in trading books and the nonnormal behaviour of most financial time series (skewness and leptokurticity).

In this paper we propose a nonparametric estimation method for copulas. The copula of multivariate distribution can be considered as the part describing its dependence structure as opposed to the behaviour of each of its margins. One attractive property of the copula is its invariance under strictly increasing transformation of the margins, and its direct link with scale invariant measures of association such as Kendall's tau and Spearman's rho. Estimation of copulas have mainly been developped in the context of bivariate i.i.d. samples either through maximum likelihood methods or nonparametric methods based on empirical distributions. In the following we use a kernel based approach. Such an approach has the advantage to provide a smooth (differentiable) reconstitution of the copula function without putting any particular parametric a priori on the dependence structure between margins. The approach is developped in the context of multivariate stationary processes satisfying strong mixing conditions. Once estimates of copulas and their derivatives are available, concepts such as positive quadrant dependence and left tail decreasing behaviour may be empirically analysed. These estimates are also useful to draw simulated data satisfying the dependence structure infered from observations. Besides, nonparametric estimators of copulas may lead to testing procedures for independence between margins. These testing procedures are in the same spirit as kernel based methods to test for serial dependence for a univariate stationary time series.

2. Framework

We consider a strictly stationary process $\{Y_t, t \in \mathbb{Z}\}$ taking values in $\mathbb{R}^n$ and assume that our data consist in a realization of $\{Y_t; t = 1, ..., T\}$. These data may correspond to observed returns of n financial assets, say stock indices, at several dates. We denote by $f(y)$, $F(y)$, the p.d.f. and c.d.f. of $Y_t = (Y_{1t}, ..., Y_{nt})'$ at point $y = (y_1, ..., y_n)'$. The joint distribution F provides complete information concerning the behaviour of Y_t. The idea behind copulas is to separate dependence and marginal behaviour of the elements constituting Y_t. The marginal p.d.f. and c.d.f. of each element Y_{jt} at point y_j, $j = 1, ..., n$, will be written $f_j(y_j)$, and $F_j(y_j)$, respectively. A copula describes how the joint distribution F is "coupled" to its univariate margins F_j, hence its name. Sklar's Theorem states that there exists an n-copula C such that for all y in $\mathbb{R}^n$,

$$F(y_1, ..., y_n) = C(F_1(y_1), ..., F_n(y_n)). \tag{1}$$

As an immediate corollary of Sklar's Theorem, we have

$$C(u_1, ..., u_n) = F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)), \tag{2}$$

where $F_1^{-1}, ..., F_n^{-1}$ are quasi inverses of $F_1, ..., F_n$ (if F_j is strictly increasing, the quasi inverse is the ordinary inverse). Copulas are thus multivariate uniform distributions which describe the dependence structure of random variables.

3. Kernel estimator of copula

From (2), we see that we are merely interested in estimating values taken by c.d.f. at distinct points, say d points in $\mathbb{R}^n$. For given $u_{ij} \in (0, 1)$, $i = 1, ..., d$, $j = 1, \ldots, n$, we assume that the c.d.f. F_j of Y_{jt}, is such that the equation $F_j(y) = u_{ij}$ admits a unique solution denoted ζ_{ij}. To build our estimators we need to introduce kernels, i.e. real bounded and symmetric functions $k_{ij}(x)$ on $\mathbb{R}$ such that

$$\int k_{ij}(x)dx = 1, \;\; i = 1, \ldots, d, \quad j = 1, \ldots, n,$$

and

$$K_i(x; h) = \Pi_{j=1}^{n} k_{ij}(x_j/h_j), \;\; i = 1, \ldots, d,$$

where the bandwidth h is a diagonal matrix with elements $(h_j)_{j=1}^n$ and determinant $|h|$ (for a scalar x, $|x|$ will denote its absolute value), while the individual bandwidths h_j are positive functions of T such that $|h| + (T|h|)^{-1} \to 0$ when $T \to \infty$. The p.d.f. of Y_{jt} at y_{ij}, i.e. $f_j(y_{ij})$, will be estimated by

$$\hat{f}_j(y_{ij}) = (Th_j)^{-1} \sum_{t=1}^{T} k_{ij}((y_{ij} - Y_{jt})/h_j),$$

while the p.d.f. of Y_t at $y_i = (y_{i1}, ..., y_{in})'$, i.e. $f(y_i)$, will be estimated by

$$\hat{f}(y_i) = (T|h|)^{-1} \sum_{t=1}^{T} K_i(y_i - Y_t; h).$$

Hence, estimators of the cumulative distribution of Y_{jt} at distinct points y_{ij} are obtained as

$$\hat{F}_j(y_{ij}) = \int_{-\infty}^{y_{ij}} \hat{f}_j(x)dx, \tag{3}$$

while estimators of the cumulative distribution of Y_t at distinct points y_i are obtained as

$$\hat{F}(y_i) = \int_{-\infty}^{y_{i1}} ... \int_{-\infty}^{y_{in}} \hat{f}(x)dx. \tag{4}$$

In order to estimate the copula at distinct points u_i, $i = 1, ..., d$, with $u_{ij} \leq u_{lj}$ for $i < l$, we use a simple plug-in method, and exploits directly expression (2):

$$\hat{C}(u_i) = \hat{F}(\hat{\zeta}_i), \tag{5}$$

where $\hat{\zeta}_i = (\hat{\zeta}_{i1}, ..., \hat{\zeta}_{in})'$ and $\hat{\zeta}_{ij} = \inf_{y \in \mathbb{R}}\{y : \hat{F}_j(y) \geq u_{ij}\}$. In fact $\hat{\zeta}_{ij}$ corresponds to a kernel estimate of the quantile of Y_{jt} with probability level u_{ij}.

The asymptotic normality of kernel estimators for copulas can be established under suitable conditions on the kernel, the asymptotic behaviour of the bandwidth, the regularity of the densities, and some mixing properties of the process.

Assumption 1 (kernel and bandwidth)
(a) Bandwidths satisfy $|h|||h||^4 T \to 0$.
(b) Kernels and bandwidths satisfy $|k_{ij}(u)| \leq c(1 + |u|)^{-(1+\omega/n)}$, *and* $||h||^{n+\omega-2} \leq c|h|$, $\omega > 2$, *for some positive finite constant* c.

Assumption 2 (process)
(a) The process (Y_t) *is strong mixing with coefficients* α_j *such that* $\sum_{l=N}^{\infty} \alpha_l = o(N^{-1})$, *as* $N \to \infty$.
(b) Second order partial derivatives for the p.d.f. of Y_t *are continuous at* ζ_i.
(c) The pdf of (Y_t, Y_{t+s}) *exists and is bounded in a neighbourhood of all pairs* (ζ_i, ζ_j), $i, j = 1, \ldots, d$, *uniformly in* $s > 1$.

Let S be the d dimensional vector with components:

$$S_i = (T|h|)^{1/2} \left\{ \hat{C}(u_i) - C(u_i) \right\},$$

and $\hat{V}$ be the d dimensional symmetric matrix such that its upper triangular matrix has components:

$$\hat{V}_{il} = \hat{C}(u_i) \int_{\mathbb{R}^n} \prod_{j=1}^{n} k_{ij}(x_j) k_{lj}(x_j) dx_j, \qquad i \leq l.$$

Proposition 1 *Under Assumptions 1 and 2, $\hat{V}^{-1/2}S$ converges in distribution to a vector of independent standard normal random variables.*

In the bivariate case $Y_t = (Y_{1t}, Y_{2t})'$, positive quadrant dependence is characterized by $C(u_1,u_2) - u_1 u_2 \geq 0$, while left tail decreasing behaviour of Y_{1t}, resp. Y_{2t}, in Y_{2t}, resp. Y_{1t}, is characterized by $C(u_1,u_2)/u_2 - \partial C(u_1,u_2)/\partial u_2 \geq 0$, resp. $C(u_1,u_2)/u_1 - \partial C(u_1,u_2)/\partial u_1 \geq 0$. We will now develop an estimator for $\partial C(u_i)/\partial u_{ij}$ based on the differentiation of $\hat{C}(u_i)$ w.r.t. u_{ij}. The estimators $\hat{C}(u_i)$ and $\partial \hat{C}(u_i)/\partial u_{ij}$ will help to detect positive quadrant dependence and left tail decreasing behaviour through the empirical counterparts of the above inequalities. Since $\dfrac{\partial \hat{C}(u_i)}{\partial u_{ij}} = \dfrac{\partial \hat{\zeta}_{ij}}{\partial u_{ij}} \dfrac{\partial \hat{F}(\hat{\zeta}_i)}{\partial \zeta_{ij}}$, by exploiting the relationship $\dfrac{\partial \hat{\zeta}_{ij}}{\partial u_{ij}} = \dfrac{1}{\hat{f}_j(\hat{\zeta}_{ij})}$, obtained through differentiation w.r.t. u_{ij} of $\int_{-\infty}^{\hat{\zeta}_{ij}} \hat{f}_j(x)dx = u_{ij}$, we get

$$\frac{\partial \hat{C}(u_i)}{\partial u_{ij}} = \frac{1}{\hat{f}_j(\hat{\zeta}_{ij})} \frac{\partial \hat{F}(\hat{\zeta}_i)}{\partial \zeta_{ij}},$$

where

$$\frac{\partial \hat{F}(\hat{\zeta}_i)}{\partial \zeta_{ij}} = \int_{-\infty}^{\hat{\zeta}_{i1}} \cdots \int_{-\infty}^{\hat{\zeta}_{iu}} \cdots \int_{-\infty}^{\hat{\zeta}_{in}} \hat{f}(x_1, ..., \hat{\zeta}_{ij}, ..., x_u, ..., x_n) \prod_{u=1, u \neq j}^{n} dx_u.$$

Let S be the d dimensional vector with components:

$$S_i = (T|h|)^{1/2} \left\{ \frac{\partial \hat{C}(u_i)}{\partial u_{ij}} - \frac{\partial C(u_i)}{\partial u_{ij}} \right\},$$

and $\hat{V}$ be the d dimensional symmetric matrix so that its upper triangular matrix has components:

$$\hat{V}_{il} = \frac{1}{\hat{f}_j(\hat{\zeta}_{ij})} \frac{\partial \hat{C}(u_i)}{\partial u_{ij}} \int_{\mathbb{R}^n} \prod_{j=1}^{n} k_{ij}(x_j) k_{lj}(x_j) dx_j, \qquad i \leq l.$$

Proposition 2 *Under Assumptions 1-2, and if $f_j(\zeta_{ij}) > 0$, $i = 1, ..., d$, $j = 1, ..., n$, $\hat{V}^{-1/2}S$ converges in distribution to a vector of independent standard normal random variables.*

Note that the extension of Proposition 2 to $\partial^k \hat{C}(u_i)/(\partial u_{i1} \partial u_{i2} ... \partial u_{ik})$, $k \leq n$, is straightforward. Such estimates are for example required for the implementation of simulation algorithms.

RESUME

Nous considérons une méthode nonparamétrique d'estimation de copules, i.e. des fonctions liant les distributions jointes à leurs marginales univariées. Nous dérivons les propriétés asymptotiques des estimateurs à noyaux des copules et de leurs dérivées dans le contexte d'un processus multivarié stationnaire satisfaisant des conditions de mélange fort.