Modelling of Default-free and Defaultable Forward LIBOR and Swap Rates

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1. Introduction

The HJM methodology is based on an exogenous specification of the dynamics of instantaneous, continuously compounded forward rates $f(t,T)$. For any fixed maturity $T \leq T^*$, the dynamics of the forward rate $f(t,T)$ are

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)dW_t,$$

where α and σ are stochastic processes, and W follows a standard Brownian motion under the real-world probability. For any maturity date $T \leq T^*$, the initial condition $f(0,T)$ is determined by the current value of the continuously compounded forward rate for the future date T which prevails at time 0. Let us denote by $B(t,T)$ the price at time $t \leq T$ of a unit zero-coupon bond which matures at the date $T \leq T^*$. The price $B(t,T)$ can be recovered from the formula

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big).$$

In an arbitrage-free setting – that is, under the martingale measure – the drift coefficient α in the dynamics of the instantaneous forward rate is uniquely determined by the volatility coefficient σ, and a stochastic process which can be interpreted as the market price of the interest-rate risk. Since the HJM approach to the term structure modelling is based on an arbitrage-free dynamics of the instantaneous continuously compounded forward rates, it requires a certain degree of smoothness with respect to maturity T of bond prices and their volatilities.

An alternative construction of an arbitrage-free family of bond prices, making no reference to the instantaneous rates, is in some circumstances more suitable. The first step in this direction was done by Sandmann and Sondermann (1993), who focused on the effective annual interest rate. This approach was further developed in ground-breaking papers by Miltersen et al. (1997) and Brace et al. (1997), who proposed to model instead the family of forward LIBOR rates. The main goal was to produce an arbitrage-free term structure model which would support the common practice of pricing such interest-rate derivatives as caps and swaptions through a suitable version of Black's formula. This practical requirement enforces the lognormality of the forward LIBOR (or swap) rate under the corresponding forward martingale measure. By the market convention, the forward LIBOR rate over accrual period $[T, T+\delta]$ is set to satisfy, for $t \leq T$,

$$L(t,T) = \frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)}.$$

The last formula makes it obvious that the volatility of the forward LIBOR rate is not deterministic if the bond price volatility follows a deterministic function. For this reason, Black's formula for caps is manifestly incompatible with the Gaussian HJM model – that is, the HJM model in which the bond price volatility $b(t,T)$ is deterministic. Consequently, the "market formula" for caps cannot be derived in this setup. It is interesting to notice that Brace et al. (1997) parametrize their version of the lognormal forward LIBOR model introduced by Miltersen et al. (1997) with a piecewise constant volatility function. They need to consider smooth volatility functions in order to analyse the model in the HJM framework, however. The backward induction approach to the modelling of forward LIBOR and swap rate, developed in Musiela and Rutkowski (1997) and Jamshidian (1997), overcomes this technical difficulty. In addition, in contrast to the previous papers, it allows also for the modelling of forward LIBOR (and swap) rates associated with accrual periods of differing lengths.

A similar approach to the modelling of market rates – referred to as the *Markov-functional approach* – was developed in a series of papers by Hunt et al. (1996, 1997) and Hunt and Kennedy (1997, 1998). Another tractable term structure model is the *rational lognormal model* proposed by Flesaker and Hughston (1996a, 1996b), and subsequently analysed by, among others, Rutkowski (1997) and Jin and Glasserman (1997).

2. Modelling of Forward LIBOR Rates

We focus on the construction of the model and the valuation of the most typical derivatives. For further details, the interested reader is referred to the original papers: Musiela and Sondermann (1993), Sandmann and Sondermann (1993), Goldys et al. (1994), Sandmann et al. (1995), Brace et al. (1997), Jamshidian (1997, 1999), Miltersen et al. (1997), Musiela and Rutkowski (1997b), Rady (1997), Brace et al. (1998), Rutkowski (1998, 1999), Glasserman and Kou (1999), and Glasserman and Zhao (2000).

3. Modelling of Forward Swap Rates

We shall first describe the most typical swap contracts and related options (the so-called *swaptions*). Subsequently, we shall present a model of forward swap rates put forward by Jamshidian (1997). For the sake of expositional convenience, we shall follow the backward induction approach due to Rutkowski (1999).

RESUME

We have presented a concise overview of recent advances in the area of term structure modelling. Special emphasis was put on the progress in the modelling of the so-called market rates, such as the forward LIBOR rates and the forward swap rates. Despite the title of the presentation, we have deliberately restricted our attention to the case of default-free contracts. Although we observe in recent years a rapidly growing interest in extending these methodologies to cover also the case of a defaultable term structure, it should be acknowledged that only partial results are available up to date. The interested reader is referred in this regard to the recent papers by Lotz and Schlögl (2000), Schönbucher (2000), as well as to the monograph by Bielecki and Rutkowski (2001).

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