Correlation Maximization under Linear and Quadratic Constraints

Akihiro HASHIMOTO
Institute of Media and Information Technology, Chiba University
1-33 Yayoi-cho, Inage-ku
Chiba 263-8522, Japan.
hashimoto@chiba-u.ac.jp

Hisao MIYANO
Faculty of Letters, Chiba University
1-33 Yayoi-cho, Inage-ku
Chiba 263-8522, Japan.
miyano@cogsci.l.chiba-u.ac.jp

Masaaki TAGURI
Faculty of Science, Chiba University
1-33 Yayoi-cho, Inage-ku
Chiba 263-8522, Japan.
taguri@math.s.chiba-u.ac.jp

1. Introduction

In our previous papers (1997, 2000), we addressed the maximization problem of correlation coefficient between a dependent variable and its linear predictor under a quadratic constraint, and proposed an algebraic method for computing the optimal linear predictor. As an extension of our method, we consider the maximization problem under quadratic and linear constraints, and derive the result that can be reduced to a maximization problem under a quadratic constraint.

2. Representation of our problem in parameter space

Let $\mathbf{z}$ be a $p-$dimensional vector of independent variables related to a dependent variable y. Then our problem is to find a weight vector $\mathbf{w}$ that maximizes the correlation between y and its linear predictor $\mathbf{z'w}$ under the constraints $\mathbf{A\,w} = \mathbf{d}$, and $(\mathbf{w} - \mathbf{w}_0)'\mathbf{B}^{-1}(\mathbf{w} - \mathbf{w}_0) \leq 1$, where $\mathbf{A,B}$ are given m by p and p by p matrices, and $\mathbf{w}_0$, $\mathbf{d}$ are given vectors, respectively.

Under the assumptions that $\mathrm{rank}(\mathbf{A}) = m < p$, and $\mathbf{B}^{-1}$ is positive, by using the singular value decomposition $(\mathbf{I}_n - \mathbf{Q}_n)\,\mathbf{Z} = \mathbf{U}$ Ë', we can obtain the representation of our problem in a parameter space, where n is the number of samples, $I_n \in R^{n \times n}$ is an identity matrix, $\mathbf{Z}$ is the design matrix, $\mathbf{Q}_n = \mathbf{11}'/n$, $\mathbf{U}$ and $\mathbf{V}$ are orthonormal matrices, and $\ddot{\mathbf{E}}$ is a diagonal matrix of the singular values. That is, our problem is equivalent to that of finding a vector $\mathbf{x}$ which maximizes $\mathbf{b}'\mathbf{x}/(\|\mathbf{b}\|\|\mathbf{x}\|)$ under the constraints $\mathbf{F}\mathbf{x} = \mathbf{d}$, and $(\mathbf{x} - \mathbf{x}_0)'\mathbf{G}^{-1}(\mathbf{x} - \mathbf{x}_0) \le 1$, where $\mathbf{x} = \ddot{\mathbf{E}}\mathbf{V}'\mathbf{w}$, $\mathbf{x}_0 = \ddot{\mathbf{E}}\mathbf{V}'\mathbf{w}_0$, $\mathbf{b} = U'\mathbf{y}$, $\mathbf{F} = \mathbf{A}\mathbf{V}\ddot{\mathbf{E}}^{-1}$, and $\mathbf{G} = (\ddot{\mathbf{E}}^{-1}\mathbf{V}'\mathbf{B}\mathbf{V}\ddot{\mathbf{E}}^{-1})^{-1}$. It is easy to see that $\mathrm{rank}(\mathbf{F}) = m$, and $\mathbf{G}^{-1}$ is positive.

Let us once again consider the singular value decomposition of the matrix $\mathbf{F}$, say $\mathbf{F} = \mathbf{U}_f \bar{\ddot{\mathbf{E}}}_f \mathbf{V}_f'$, where matrices $\mathbf{U}_f$ and $\mathbf{V}_f$ are orthonormal, and $\bar{\ddot{\mathbf{E}}}_f = (\ddot{\mathbf{E}}_f, \mathbf{O})$ where $\ddot{\mathbf{E}}_f$ is a diagonal matrix of m nonzero singular values, and $\mathbf{O}$ is a m by (p-m) zero matrix. Then we find that our problem equals the maximization problem under a quadratic constraint; that is, our problem is to find a vector $\mathbf{u}$ that maximizes $\mathbf{h}'\mathbf{u}/(\|\mathbf{h}\|\|\mathbf{u}\|)$ under the constraint $(\mathbf{u} - \mathbf{u}_0)'\mathbf{H}^{-1}(\mathbf{u} - \mathbf{u}_0) \le 1$, where $\mathbf{h}$ and $\mathbf{u}_0$ are vectors of last ($p - m$) elements of $\mathbf{V}_f'\mathbf{b}$ and $\mathbf{V}_f'\mathbf{x}_0$ respectively, $\mathbf{H}^{-1}$ is a matrix determined by $\mathbf{U}_f$, $\mathbf{V}_f$, $\ddot{\mathbf{E}}_f$ and $\mathbf{G}^{-1}$. The optimal $\mathbf{x}$ is given by $\mathbf{x}' = (\mathbf{d}'\mathbf{U}_f\ddot{\mathbf{E}}_f^{-1}, \mathbf{u}')\mathbf{V}_f'$. On the computational algorithm for the optimal $\mathbf{u}$, see Hashimoto, Miyano & Taguri (2000).

Rigorous derivation of the above result and numerical examples will be given at the meeting.

REFERENCES

Hashimoto, A., Miyano, H. and Taguri, M. (2000). On optimal solutions under quadratic constrains. Proc. of 10th Japanese and Korean Joint Conference of Statistics, 203-208.

Hashimoto, A., Miyano, H. and Taguri, M. (1997). Maximization of correlation coefficient under quadratic constrains. Proc. of 6th China and Japan Symposium on Statistics, 64-67.

RESUME

Dans nos articles précédents (1997, 2000), nous avons traité le problème de la maximisation du coefficient de corrélation entre une variable dépendante et son prédicteur linéaire sous une contrainte quadratique et nous avons proposé une méthode algébrique pour calculer le prédicteur linéaire optimal. Comme extension de notre méthode, nous considérons le problème de la maximisation sous des contraintes quadratiques et linéaires et nous en tirons le résultat pouvant être réduit à un problème de maximisation sous une contrainte quadratique.