Estimation and forecasting a time varying spectral density function by using a locally stationary autoregressive model

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1. Introduction

In various aspects of data analysis on natural phenomenon, it is often required to forecast future changes of nonstationary time series data, whose statistical structure changes slowly. Our concern in this study is time-varying spectral density function of the nonstationary process whose statistical structure changes gradually over time. In this paper, we propose the method to estimate and forecast time varying spectral density function, which is based on a locally stationary autoregressive model. And then, the validity of our method is shown from the standpoint of forecasting performance, by means of numerical experiments.

2. A Method for forecasting time varying spectrum

Let $\{X_t\}$ follows the nonstationary oscillatory process (for example, Priestley(1981)) whose mean is zero, and satisfies an autoregressive model of order p with time-varying coefficients

$$X_t - \beta_{1,t}X_{t-1} - \cdots - \beta_{p,t}X_{t-p} = \sigma_t\varepsilon_t \tag{1}$$

where ε_t is the random variable which follows the white noise process with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = 1$. Here, the coefficients $(\beta_{1,t}, \ldots, \beta_{p,t}, \sigma_t)$ are assumed to be the continuous functions whose values change slowly over t. In this paper, we focus on the changes of the parameters for every short time interval $[(n-1)M, nM]$, where M is unknown, and consider the changes with repect to n. Now we consider the spectral density function at n,

$$f(\lambda, n) = \frac{\sigma_n^2}{|1 + \beta_{1,n}e^{-i2\pi\lambda} + \cdots + \beta_{p,n}e^{-i2p\pi\lambda}|^2} \tag{2}$$

Here, for the estimation of the parameters $(\beta_{1,n}, \ldots, \beta_{p,n}, \sigma_n^2)$, we use M samples, $\{X_t; t \in [(n-1)M+1, nM]\}$. We explain the dynamic relationship among these parameters by the following multivariate autoregressive model,

$$\boldsymbol{\theta}_n = A_1\boldsymbol{\theta}_{n-1} + A_2\boldsymbol{\theta}_{n-2} + \ldots + A_K\boldsymbol{\theta}_{n-K} + \boldsymbol{\delta}_n \tag{3}$$

where $\boldsymbol{\theta}_n = (\beta_{1,n}, \ldots, \beta_{p,n}, \sigma_n^2)'$ (where the symbol $'$ means transposition), K is an unknown order, A_k ($k = 1, \ldots, K$) are unknown coefficient matrices, and $\boldsymbol{\delta}_n$ is a vector of white noise. To forecast the future values $\boldsymbol{\theta}_{n+l}$ ($l = 1, \ldots, L$), we use the linear predictor $\tilde{\boldsymbol{\theta}}_{n+l}$, which is defined by

$$\tilde{\boldsymbol{\theta}}_{n+l} = \tilde{A}_1\boldsymbol{z}_{n+l-1} + \tilde{A}_2\boldsymbol{z}_{n+l-2} + \ldots + \tilde{A}_K\boldsymbol{z}_{n+l-K}$$

where $\boldsymbol{z}_{n+l-m} = \boldsymbol{\theta}_{n+l-k}$ $(l \leq k)$ and $\boldsymbol{z}_{n+l-m} = \tilde{\boldsymbol{\theta}}_{n+l-k}$ (otherwise). Also, $\{\tilde{A}_i\}$ are the least squares estimates of the coefficient matrices. A predictor on spectral density function at l steps ahead can be defined by

$$\tilde{f}(\lambda, n+l) = \frac{\tilde{\sigma}^2_{n+l}}{|1 + \tilde{\beta}_{1,n+l}e^{-i2\pi\lambda} + \cdots + \tilde{\beta}_{p,n+l}e^{-i2p\pi\lambda}|^2}$$

where $(\tilde{\beta}_{1,n+l}, \ldots, \tilde{\beta}_{p,n+l}, \tilde{\sigma}^2_{n+l})$ are values forecasted at $n+l$, which is obtained by multi-step forecasting of $\boldsymbol{\theta}_{n+l}$.

Finally, we show the method to choose unknown time interval and the orders of the models (1) and (3). The choice of the orders, p, K, and the time interval M affect to the forecasting accuacy of $\tilde{\boldsymbol{\theta}}_{n+l}$, because the number of parameters to be estimated and the sample size to use for estimation differs depending on their values and then it has the possibility to give negative effects on forecasting spectrum, depending on the estimation accuracies of coefficient matrices $\{A_i\}$. We choose their values such that the sum of squared forecasting errors over the frequency λ,

$$S(p(l), K(l), M(l)) = \sum_{j=1}^{n-l} \int_{\lambda} (\hat{f}(\lambda, j|M(l)) - \tilde{f}(\lambda, j|p(l), K(l), M(l)))^2 d\lambda$$

is minimized for every forecast step l, where $p(l), K(l)$ and $M(l)$ are the values of p, K and M under l, respectively, $\hat{f}(\lambda, j|M(l))$ is the estimates obtained by using the time series data $\{X_t; t \in [(j-1)M(l)+1, jM(l)]\}$ and $\tilde{f}(\lambda, j|p(l), K(l), M(l))$ is the forecasted values under $p(l), K(l)$ and $M(l)$ are fixed.

3. Numerical results on forecasting accuracy

As a practial example, we show numerical results of the accuracies on forecasting time varying spectral density function up to 10 steps ahead, when we used a measured time series data on the sea surface movement in the wave developing process. Table 1 shows the mean values of squared forecasting errors obtained by experiments of 1350 times, when we applied A) a time invariant predictor over forecasting period, B) the time varying predictor which is not taken into account of the dependency among the parameters $(\beta_{1,n}, \ldots, \beta_{p,n}, \sigma^2_n)$ and C) our predictor. We can admit that our predicor gives the smallest values in the three predictors and it is evaluated that the predictor is effective from the numerical standpoint(The details are shown in the presentation).

Table 1. Numerical Comparisons of Mean Squared Forecasting Errors

Step	(A)	(B)	(C)
1	110.4	90.5	36.1
2	212.9	149.0	58.7
3	248.3	122.8	67.7
4	265.9	108.0	75.0
5	329.6	120.3	96.3
6	342.9	137.8	112.1
7	340.8	131.5	115.9
8	412.0	182.6	166.1
9	595.7	425.8	259.1
10	764.1	639.6	317.7

REFERENCES

Priestley, M.B. (1981). *Spectral Analysis and Time Series. Vol.2. Academic Press.*

RÉSUMÉ

Cet article concerne le spectre non-stationnaire. Nous proposons une méthode pour prévoir les changements de la densité spectrale en série chronologique non-stationnaire. La validité de la méthode proposée isi sera démontrée au travers de l'essai de son applicabilité aux données de la surface de la mar qui montre des états non-stationaires.