A New Spectral Analysis in Time Series Data

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1. Introduction

The classical Fourier analysis are used to detect periodic trends that are of the sinusoidal shape in time series data. The basic idea of the Fourier series is that a time series data \( x(t) \) can be decomposed into a sum of sines and cosines:

\[
x(t) = \sum_{k=0}^{\infty} a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T}.
\]

This composition shows that \( x(t) \) is a sum of sinusoidal shapes at frequencies \( \lambda_k = 2\pi k / T \) for \( k = 0, 1, \ldots \). In addition, the variability in \( x(t) \) as measured by \( \int_0^T |x(t)|^2 \, dt \) partitions into the sum of the variabilities of the sinusoidal shapes. The Fourier analysis is to treat the partition as an analysis of variance (ANOVA) for identifying sinusoidal periodicities in a time series data \( x(t) \).

This article describes an ANOVA technique that is better suited for identifying non-sinusoidal periodic components. In addition, the new technique can be used to investigate the shapes of these periodic components.

2. A New Spectral Analysis

Let \( L^2 \) be the Hilbert space with the inner product \( < x, y >_L = \frac{1}{2\pi} \int_0^{2\pi} x(t) y(t) \, dt \) and let \( \lfloor b \rfloor \) be the largest integer no larger than \( b \), for example, \( \lfloor 3.5 \rfloor = 3 \) and \( \lfloor -3.5 \rfloor = -4 \).

The complex-valued functions

\[
f_{k,n,j}(t) = \exp\left(\frac{\pi}{n}\left[\frac{2kt}{\pi} + \frac{2}{n} - \frac{j-1}{n} + 1\right]\right) \quad \text{and} \quad f_{-k,n,j}(t) = \exp\left(\frac{\pi}{n}\left[-\frac{2kt}{\pi} + \frac{2}{n} - \frac{j-1}{n}\right]\right)
\]

are step functions that takes the values \(-1, 0, 1\). The functions \( f_{k,n,j} \) and \( f_{-k,n,j} \) are conjugates of each
other. Construct $B_{k,n}$ to be the subspace of $L^2$ spanned by $f_{k,n,1}, \ldots, f_{k,n,n}$. The direct sum $L_{m,n} = B_{-m,n} + \ldots + B_{m,n}$ is the subspace of the of $L^2$ spanned by the functions $\{f_{k,n,j}(t)\}$. For $x(t), y(t)$ in $L_{m,n}$, define the inner product $\langle x, y \rangle_G = \int_0^1 x_u y_u d\nu$, where $x_u(t) = \sum_{k=-m}^m x_k(t + \frac{4\mu \pi}{2k})$ for the time shift parameter $4\mu \pi / 2k$.

The central result follows.

**THEOREM 1.** The subspaces $B_{-m,n}, \ldots, B_{m,n}$ of $L_{m,n}$ are mutually orthogonal with respect to the inner product $\langle x, y \rangle_G$. Furthermore, each function $x(t)$ in $L^2$ has the unique new spectral representation

$$x(t) = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{k=-m}^m P_G(x_{m,n}B_{k,n})(t)$$

(2)

where $x_{m,n}(t)$ is the projection $P_L(x|L_{m,n})(t)$ of $x(t)$ onto $L_{m,n}$ with respect to the inner product $\langle x, y \rangle_L$ and where $P_G(x_{m,n}|B_{k,n})(t)$ is the projection of $x_{m,n}(t)$ onto $B_{k,n}$ with respect to the inner product $\langle x, y \rangle_G$. The component $P_G(x_{m,n}|B_{k,n})(t)$ in (2) is periodic with period $2\pi / k$ and it may take non-sinusoidal shapes.

The squared norm of this series approximation is

$$\left| \sum_{k=-m}^m P_G(x_{m,n}|B_{k,n})(t) \right|_G^2 = \sum_{k=-m}^m \left| P_G(x_{m,n}|B_{k,n})(t) \right|_L^2$$

(3)

The ANOVA partition in (3) is the basis for the new spectral analysis.

**REFERENCE**


