Likelihood Ratio Test for Composite Hypotheses Based on Multivariate Normal Response

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1 Introduction

Kudo (1963) proposed a one-sided test for testing the superiority of one treatment of two treatments based on multivariate normal response with known covariance matrix by using likelihood ratio method. But it is difficult to use the method practically because of its theoretical complications. Thus O'Brien (1984) derived a one-sided test with an alternative hypothesis that the superiority of one treatment is expressed by a fixed vector. And Tang *et al.* (1989) considered a one-sided test generalizing O'Brien (1984).

In this study we consider a one-sided test and a two-sided test for testing the superiority and the inferiority between two treatments based on a multivariate normal response with unknown covariance matrix. First we derive a statistic by using the likelihood ratio method of our composite hypotheses. Based on the statistic, we determine the critical value by using the conditional distribution considered by Wang-McDermott (1998). Since the power of our test depends on the unknown covariance matrix, we investigate how the power is influenced by covariance matrix through the simulations.

2 Likelihood ratio test for composite hypotheses

We consider two treatments A and B with p-variate responses denoted by random variable vectors $\boldsymbol{X}_A$ and $\boldsymbol{X}_B$ respectively. We suppose that $\boldsymbol{X}_A$ and $\boldsymbol{X}_B$ are independently distributed according to p-variate normal $N(\boldsymbol{\mu}_A, \boldsymbol{\Sigma})$ and $N(\boldsymbol{\mu}_B, \boldsymbol{\Sigma})$ respectively with mean vectors $\boldsymbol{\mu}_A$ and $\boldsymbol{\mu}_B$ and an unknown covariance matrix $\boldsymbol{\Sigma}$. We assume $\boldsymbol{\mu}_A - \boldsymbol{\mu}_B = \lambda\boldsymbol{\delta}$ where λ is an unknown scalar and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)'$ is a known p-dimensional vector satisfying $\delta_1 \geq 0, \cdots, \delta_p \geq 0$ with strict inequality for either one. Here we test the null hypothesis H_0 against alternative hypothesis H_1 as (O-S) $H_0 : \lambda = 0,\ H_1 : \lambda > 0$ and (T-S) $H_0 : \lambda = 0,\ H_1 : \lambda \neq 0$. Thus (O-S) is a one-sided test and (T-S) is a two-sided test. We derive the likelihood ratio test statistic of our composite hypotheses. If we let $\boldsymbol{X} = (\boldsymbol{X}_A - \boldsymbol{X}_B)/\sqrt{2}$, $\boldsymbol{X}$ is distributed according to $N(\lambda\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = \boldsymbol{\delta}/\sqrt{2}$. Let L be the likelihood function based on a sample $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ from $N(\lambda\boldsymbol{\mu}, \boldsymbol{\Sigma})$. When we fix λ, $\boldsymbol{\Sigma}$ which maximizes L is equal to $\{\boldsymbol{S} + n(\bar{\boldsymbol{X}} - \lambda\boldsymbol{\mu})(\bar{\boldsymbol{X}} - \lambda\boldsymbol{\mu})'\}/n$. Then $\hat{\lambda} = 0$ under H_0 where $\hat{\lambda}$ is the maximum likelihood estimator of λ. Under $H_0 \cup H_1$ in (O-S), $\hat{\lambda} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\bar{\boldsymbol{X}}/\boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu}$ if $\boldsymbol{\mu}'\boldsymbol{S}^{-1}\bar{\boldsymbol{X}} \geq 0$, and $\hat{\lambda} = 0$ otherwise. $\hat{\lambda} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\bar{\boldsymbol{X}}/\boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu}$ under $H_0 \cup H_1$ in (T-S). By using these maximum likelihood estimators, we obtain the likelihood ratio statistic $\boldsymbol{T} = \boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{Z}/\sqrt{\boldsymbol{\mu}'\boldsymbol{S}^{-1}\boldsymbol{\mu}(1 + \boldsymbol{Z}'\boldsymbol{S}^{-1}\boldsymbol{Z})}$ where $\boldsymbol{Z} = \sqrt{n}\bar{\boldsymbol{X}}$. Thus given a critical value c the rejection region of H_0 is given by $\boldsymbol{T} > c$ for (O-S), and $|\boldsymbol{T}| > c$ for (T-S).

We determine c for a specified significance level α. Since the explicit distribution of $\boldsymbol{T}$ does not exist, we determine c by using the distribution of $\boldsymbol{Z}$. Then we use the conditional distribution of $\boldsymbol{Z}$ given the value of $\boldsymbol{V} = \sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{X}_i^{'}$ which is the complete and sufficient statistic for $\boldsymbol{\Sigma}$ under H_0 to determine c independent of unknown $\boldsymbol{\Sigma}$ (see Wang-McDermott (1998)). The conditional density of $\boldsymbol{Z}$ given $\boldsymbol{V} = \boldsymbol{v}$ under H_0 is $f_0(\boldsymbol{z}|\boldsymbol{v}) = k_0|\boldsymbol{v} - \boldsymbol{z}\boldsymbol{z}^{'}|^{\frac{n-p-2}{2}} I_{\boldsymbol{\Omega}}(\boldsymbol{z})$ where k_0 is constant, $\boldsymbol{\Omega} = \{\boldsymbol{z} \in \boldsymbol{R}^p : \boldsymbol{v} - \boldsymbol{z}\boldsymbol{z}^{'}$ is positive definite$\}$ and $I_{\boldsymbol{\Omega}}(\boldsymbol{z})$ is the function satisfying $I_{\boldsymbol{\Omega}}(\boldsymbol{z}) = 1$ for $\boldsymbol{z} \in \boldsymbol{\Omega}$ and $I_{\boldsymbol{\Omega}}(\boldsymbol{z}) = 0$ for $\boldsymbol{z} \notin \boldsymbol{\Omega}$. Thus the probability that H_0 is rejected given $\boldsymbol{V} = \boldsymbol{v}$ under H_0 is $\int \cdots \int_{\boldsymbol{D}(c) \cap \boldsymbol{\Omega}} f_0(\boldsymbol{z}|\boldsymbol{v}) dz_1 \cdots dz_p$ where $\boldsymbol{D}(c)= \{\boldsymbol{z} \in \boldsymbol{R}^p : \boldsymbol{T} > c\}$ for (O-S) and $\boldsymbol{D}(c)= \{\boldsymbol{z} \in \boldsymbol{R}^p : |\boldsymbol{T}| > c\}$ for (T-S). So we determine c such that this integration may be equal to α.

Next we consider the power of test under $H_1 F \lambda = \lambda_1$ for a fixed λ_1 ($\lambda_1 > 0$ for (O-S) and $\lambda_1 \neq 0$ for (T-S)), although we must consider all values of λ under H_1. The conditional density of $\boldsymbol{Z}$ given $\boldsymbol{V} = \boldsymbol{v}$ under H_1 is $f_1(\boldsymbol{z}|\boldsymbol{v}) = k_1|\boldsymbol{v} - \boldsymbol{z}\boldsymbol{z}^{'}|^{\frac{n-p-2}{2}} \exp(\lambda_1 \sqrt{n} \boldsymbol{\mu}^{'} \boldsymbol{\Sigma}^{-1} \boldsymbol{z}) I_{\boldsymbol{\Omega}}(\boldsymbol{z})$ where k_1 is constant. Since $\boldsymbol{V}$ is not sufficient for $\boldsymbol{\Sigma}$ under H_1, the conditional distribution of $\boldsymbol{Z}$ given $\boldsymbol{V}$ depends on $\boldsymbol{\Sigma}$. Then $\pi(\lambda_1, \boldsymbol{\Sigma})= \int \cdots \int_{\boldsymbol{D}(c)} f_1(\boldsymbol{z}|\boldsymbol{v}) dz_1 \cdots dz_p$ is the power when the covariance matrix is equal to $\boldsymbol{\Sigma}$.

3 Simulations

Suppose $p = 2$, $\boldsymbol{\mu} = (1, 1)^{'}$, $n = 30$, $\alpha = 0.05$ and $\boldsymbol{v} = \begin{pmatrix} 42.0 & 3.0 \\ 3.0 & 24.0 \end{pmatrix}$. Then $c = 0.413$ for (O-S) and $c = 0.479$ for (T-S). Table 1 gives $\pi(\lambda_1, \boldsymbol{\Sigma})$ for $\lambda_1 = 0.3,\ 0.6$ where $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $\rho = -0.9,\ -0.6,\ -0.3,\ 0,\ 0.3,\ 0.6,\ 0.9$. It shows that $\pi(\lambda_1, \boldsymbol{\Sigma})$ decreases as ρ increases from -0.9 to 0.9 for each λ_1 and $\pi(\lambda_1, \boldsymbol{\Sigma})$ increases as λ_1 increases from 0.3 to 0.6 for each ρ in (O-S) and (T-S). And $\pi(\lambda_1, \boldsymbol{\Sigma})$ in (O-S) is higher than $\pi(\lambda_1, \boldsymbol{\Sigma})$ in (T-S) for each ρ and λ_1.

Table 1FPower of test $\pi(\lambda_1, \boldsymbol{\Sigma})$

		ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
(O-S)	λ_1@	0.3	0.999	0.999	0.974	0.862	0.713	0.580	0.476
		0.6	0.999	0.999	0.999	0.998	0.984	0.946	0.886
(T-S)	λ_1@	0.3	0.999	0.999	0.937	0.761	0.578	0.439	0.342
	@@	0.6	0.999	0.999	0.999	0.992	0.957	0.887	0.794

REFERENCES

Kudo, A. (1963). A Multivariate Analogue of the One-sided Test. *Biometrika* **50**, 403-418.

O'Brien, P.C. (1984). Procedures for Comparing Samples with Multiple Endpoints, *Biometrics* **40**, 1079-1087.

Tang, D.-I., Gnecco, C. and Geller, N.L. (1989). An Approximate Likelihood Ratio Test for a Normal Mean Vector with Nonnegative Components with Application to Clinical Trials, *Biometrika* **76**, 577-583.

Wang, Y. and McDermott, M.P. (1998). Conditional Likelihood Ratio Test for a Nonnegative Normal Mean Vector, *Journal of the American Statistical Association* **93**, 380-386.