Sequential Confidence Intervals for a Function of the Parameters of a Negative Exponential Distributrion

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1. Introduction

Let $X_1, X_2, X_3, \cdots$ be independent and identically distributed random variables according to a negative exponential distribution with location parameter $\mu \in (-\infty, \infty)$ and scale parameter $\sigma \in (0, \infty)$ both unknown. Its probability density function is given by

$$f_{\mu,\sigma}(x) = \begin{cases} \sigma^{-1} \exp\{-(x-\mu)/\sigma\} & \text{for} \quad \mu < x < \infty \\ 0 & \text{for} \quad -\infty < x \leq \mu. \end{cases}$$

We want to construct a confidence interval of a function $\theta = \theta(\mu, \sigma)$ with any given confidence coefficient $1 - \alpha \in (0, 1)$ and interval length $2d \in (0, \infty)$. Our aim is to find an appropriate sample size which satisfies the condition on the confidence interval. Let $T_n = \min\{X_1, \cdots, X_n\}$, $\sigma_n = (n-1)^{-1} \sum_{i=1}^n (X_i - T_n)$, $\theta_n = \theta(T_n, \sigma_n)$ and an interval $I_n = [\,\theta_n - d,\, \theta_n + d\,]$. Here we use θ_n as an estimator of θ and impose the following assumption.

Assumption (A)

$\theta(x, y)$ has continuous partial derivatives $\dfrac{\partial \theta(x,y)}{\partial x}$ and $\dfrac{\partial \theta(x,y)}{\partial y}$ on $R \times R_+$, and

$$\frac{\partial \theta(\mu,\sigma)}{\partial y} \equiv \left. \frac{\partial \theta(x,y)}{\partial y} \right|_{(x,y)=(\mu,\sigma)} \neq 0.$$

Let $\tau^2 = \sigma^2 \left\{ \dfrac{\partial \theta(\mu,\sigma)}{\partial y} \right\}^2 (> 0)$. Since $\sqrt{n-1}\,(\theta_n - \theta) \to N(0, \tau^2)$ as $n \to \infty$ in distribution, we get that $P\{|\theta_n - \theta| \leq d\} \approx P\{|N(0,1)| \leq \frac{d}{\tau}\sqrt{n-1}\}$ as $n \to \infty$. Choose $u = u_\alpha > 0$ such that $\Phi(u) - \Phi(-u) = 1 - \alpha$, where Φ is the standard normal distribution function. From the above approximation result one can find an asymptotic optimal fixed sample size n^* for sufficiently small $d > 0$ as follows. Let

$$n^* = u^2 \tau^2 d^{-2}.$$

Clearly

$$\frac{d}{\tau}\sqrt{n-1} \geq u \quad \Longleftrightarrow \quad n \geq u^2 \tau^2 d^{-2} + 1 = n^* + 1.$$

Here n^* is assumed to be an integer for simplicity. Since μ and σ are unknown, n^* is also unknown. Thus we will try to solve this problem sequentially.

2. Main results

Motivated by the form of n^* we propose the following stopping rule.

$$(1) \quad N = N(d) = \inf\{n \geq m : n \geq u^2 \tau_n^2 d^{-2} + 1\},$$

where $m(\geq 2)$ is a starting sample size and $\tau_n^2 = \sigma_n^2 \left\{ \dfrac{\partial \theta(T_n, \sigma_n)}{\partial y} \right\}^2$. Then $P(N(d) < \infty) = 1$ for all $d > 0$. Once the sampling stops, we use I_N as a confidence interval of θ. Then, by making use of the fact that $\sqrt{N-1}\,(\theta_N - \theta) \to N(0, \tau^2)$ as $d \to 0$ in distribution under Assumption (A) we can obtain the asymptotic consistency of the sequential confidence intervals $\{I_N,\, d > 0\}$ in the sense of Chow and Robbins.

Theorem 1. (asymptotic consistency) Under Assumption (A)

$$P(\theta \in I_N) \to 1 - \alpha \quad \text{as } d \to 0.$$

Example of θ

We shall now give an example of θ. Let $\theta(x, y) = \dfrac{x}{y}$. Hence we want to construct a confidence interval of $\theta = \dfrac{\mu}{\sigma}$. Then $\tau^2 = \dfrac{\mu^2}{\sigma^2}$, $\theta_n = \dfrac{T_n}{\sigma_n}$ and $\tau_n^2 = \dfrac{T_n^2}{\sigma_n^2}$. Let N be defined by (1). Due to the results of Aras and Woodroofe (1993) we have the following theorem which provides the asymptotic consistency of the sequential confidence intervals $\{I_N,\, d > 0\}$ for $\dfrac{\mu}{\sigma}$ and the second order asymptotic expansion of the expected sample size $E(N)$.

Theorem 2. Let $0 < \mu < \infty$. Then under Assumption (A)

$$P(\theta \in I_N) \to 1 - \alpha \quad \text{as } d \to 0$$

and

$$E(N) = n^* + \rho + \frac{2\sigma}{\mu} + o(1) \quad \text{as } d \to 0,$$

where ρ is a constant satisfying $0 < \rho < \frac{5}{2}$.

REFERENCE

Aras, G. and Woodroofe, M. (1993). Asymptotic expansions for the moments of a randomly stopped average. Ann. Statist. 21, 503–519.