Maximum Likelihood Predictive Densities for the Length-biased Inverse Gaussian Distribution

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1. SUMMARY

This article deals with the estimation of parameters of a length-biased inverse Gaussian distribution via maximum predictive density estimation.

2. INTRODUCTION

A length-biased distribution arises as a selection model where the sampling distribution follows a weighted version of the original population due to a stochastic mechanism which distorts the probability of an item being selected. When observations are selected with probability proportional to their “length”, the resulting distribution is called *length-biased*. A distribution function \( G_F \) defined on \( R^+ \) is called *length-biased* distribution corresponding to a df \( F \) (also defined on \( R^+ \)), if

\[
G_F(y) = \mu_F^{-1}\left\{ \int_{0}^{y} x \cdot dF(x) \right\}, \quad \forall y \in R^+,
\]

where \( \mu_F = \int_{0}^{\infty} x \cdot dF(x) \).

Thus, the length-biased pdf can be written as

\[
g(x) = \frac{x \cdot f(x)}{E(X)}
\]

**Predictive Density Function**

Let \( X_1, \ldots, X_n \) be a sample of past observations, and let \( T \) be a future observation from a population with pdf \( f(x; \theta) \). Then the MLPD for \( T = t \) based on \( X = x \) defined as

\[
\hat{f}(t|x) \propto \max_{\theta} f(t; \theta) f(x; \theta).
\]
In the case of length-biased inverse Gaussian random variable, the joint pdf

\[
 f(t; \mu, \lambda)f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi \mu^2}\right)^{\frac{n+1}{2}} t^{-\frac{n}{2}} \prod_{i=1}^{n} x_i^{-\frac{1}{2}} \exp\left\{-\frac{\lambda}{2\mu^2} \sum \frac{(x_i - \mu)^2}{x_i}\right\} \\
 \cdot \exp\left\{-\frac{\lambda}{2\mu^2} \frac{(t - \mu)^2}{t}\right\}
\]  

(1)

The maximum of the joint pdf with respect to \( \mu \), or \( \lambda \) (or w.r.t. both) provides us with (restricted or unrestricted) MLPD's for \( t \).

**Estimation of Reliability**

The reliability function is defined as \( R(t_0) = 1 - F(t_0) \) where \( F(x) = \int_{0}^{x} f(y)dy \). The reliability function of LBIG is given by

\[
 \Phi(-\alpha(t_0)) + \exp(2\lambda/\mu)\Phi(\beta(t_0))
\]

where \( \alpha(t_0) = \sqrt{\lambda/t_0(t_0/\mu - 1)} \), \( \beta(t_0) = -\sqrt{\lambda/t_0(t_0/\mu + 1)} \), and \( \Phi(.) \) is the cdf of standard normal random variable. In the sense of maximum likelihood estimation, one would replace the parameters with their MLE’s to obtain an estimate of the reliability at a fixed time point. In the case of MLPD estimation, we proceed by replacing the pdf with its MLPD estimate to obtain an estimate of the reliability. The MLPD of \( R(t|\theta) = 1 - \int_{0}^{t} \tilde{f}(y|x)dy \) where \( \tilde{f}(.) \) is an MLPD of \( f(.) \).

**REFERENCES**


**RESUME**

Cet article traite l’évaluation des paramètres d’une distribution gaussienne inverse longueur-décentre par l’intermédiaire d’évaluation prédicitive maximum de densité.