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Moments of the Maximum (Minimum) of Bivariate Normal Random Variables

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1. Relation between the moment of maximum and minimum

Clark(1961) gave two applications of max and min of dependent variables in operational research, while Berry, Evett and Pinchin(1992) indicate their relevance in statistics of DNA matching. We shall show that the moments of max(min) can be found from min(max).
Consider a bivariate r.v $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and let

$$f_{i,j}(z) = (1/\sigma_i)\phi[(z-\mu_i)/\sigma_i]\Phi[\frac{(z-\mu_j)/\sigma_j - \rho(z-\mu_i)/\sigma_i}{(1-\rho^2)^{1/2}}], \;\; i \neq j, \;\; i,j = 1,2$$

where ϕ and Φ respectively denote standardized univariate normal density and its distribution function. It is then easy to see that the probability density function(p.d.f) of $\max(X_1, X_2)$ is $g(z) = f_{12}(z) + f_{21}(z), \;\; z \in \mathcal{R}$ and that it is related to $h(z)$, the p.d.f of $\min(X_1, X_2)$ as shown below

$$h(z) = (1/\sigma_1)\cdot\phi[(z-\mu_1)/\sigma_1)] + (1/\sigma_2)\cdot\phi[(z-\mu_2)/\sigma_2)] - g(z), \;\; z \in \mathcal{R} \tag{1}$$

Expressions of $g(z)$ and $h(z)$ simplify in case $X_i, i = 1,2$ are identically distributed and additionally if they are also independent. Next if we let m_k and M_k denote respectively $k-$th raw moment of the $\min(X_1, X_2)$ and the $\max(X_1, X_2)$, then from (1) we have

$$m_k = \sum_{j=1}^{2} \mu'_{kj} - M_k, \;\; k \geq 0 \tag{2}$$

where μ'_{kj} is the $k-$ moment of $X_j \sim N(\mu_j, \sigma_j^2), \;\; j = 1,2$. This shows that the moments of the max(min), can be derived from the moments of the min(max). Substituting in (2), the first two moments of max from Clark(1961) and simplifying, we have the first two moments of min in Cain (1994) obtained through moment generating function as

$$m_1 = \mu_1\Phi(-\theta) + \mu_2\Phi(\theta) - a\phi(\theta) \text{ and} m_2 = (\mu_1^2+\sigma_1^2)\Phi(-\theta) + (\mu_2^2+\sigma_2^2)\Phi(\theta) - (\mu_1+\mu_2)\phi(\theta),$$

where $\theta = (\mu_1 - \mu_2)/a, a^2 = Var(X_1 - X_2)$.
Clark(1961) provides first four moments of the max and further his results are easily extendable to the higher order moments. The determination of higher moments of min from moment generating function is cumbersome.

2. Moments of max and min from those of Range and mid range

Here we show that the moments of max and min can be found from those of range and the mid range and additionally this approach provides their mixed moments without resorting to their joint distribution.
For a bivariate r.v. we know

$$\max(X_1, X_2) = 1/2(X_1 + X_2 + \mid X_1 - X_2 \mid), \text{and} \min(X_1, X_2) = 1/2(X_1 + X_2 - \mid X_1 - X_2 \mid)$$

that is

$$\max(X_1, X_2) - \min(X_1, X_2) = \mid X_1 - X_2 \mid = R, \;\; \max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2 = Q, \tag{3}$$

where R and Q respectively is the range and twice the mid range. The moments of r.v $Q \sim N(\nu_2, \tau^2)$, where $\nu_2 = \mu_2 + \mu_1$ and $\tau_2^2 = \sigma_2^2(1 + \tau^2 + 2\tau\rho), \;\; \tau = \sigma_1/\sigma_2$, need no new computations. The r.v $X_2 - X_1 \sim N(\nu_1, \tau_1^2)$, where $\nu_1 = \mu_2 - \mu_1, \;\; \tau_1^2 = \sigma_2^2(1 + \tau^2 - 2\tau\rho)$ leads to the p.d.f of R as

$$f_R(r) = (1/\tau_1)\phi[(r - \nu_1)/\tau_1] + (1/\tau_1)\phi[(r + \nu_1)/\tau_1], \;\; r \geq 0.$$

and its k-th moment $ER^k = \int_{-\nu_1/\tau_1}^{\infty} (\tau_1 \cdot z + \nu_1)^k \phi(z)dz + \int_{\nu_1/\tau_1}^{\infty} (\tau_1 \cdot z - \nu_1)^i \phi(z)dz$

can be evaluated, first expanding binomial term in the integrand, then replacing z^i by Hermite polynomials $H_i, i = 0, 1, 2, \ldots, k$ and finally solving it with known result $\int_m^\infty H_i(z)\phi(z) = -H_{i-1}(m)\phi(m), i = 1, 2, ..., k$. In particular we have

$$ER = 2\tau_1\phi(\nu_1/\tau_1) + \nu_1[\Phi(\nu_1/\tau_1) - \Phi(-\nu_1/\tau_1)] \text{ and } ER^2 = (\tau_1^2 + \nu_1^2)[\Phi(\nu_1/\tau_1) + \Phi(-\nu_1/\tau_1)].$$

Expressions for the higher moments are lengthy but straightforward. We claim that the moments of max and min can be found from $\gamma_{i,k-i} = E(R^iQ^{k-i}),\ \ i = 0, 1, 2, \ldots, k$. We treat separately the following two cases
a) r.vs. Q and R are independent; *b*) correlated.
a) Now the normally distributed r.vs. $X_2 - X_1$ and Q have $Cov(Q, X_2 - X_1) = \sigma_2^2 - \sigma_1^2 = \sigma_2^2(1 - \tau^2)$. Consequently are independente iff $\tau = 1$, i.e, iff $X_i, i = 1, 2$ have equal variance. Note that even if $\rho = 0$, i.e, $X_i, i = 1, 2$ are independent, the r.v Q and R are correlated. From (3), we have a linear system of $k + 1$ equations

$$\gamma_{i,0}\gamma_{0,k-i} = \gamma_{i,k-i} = \sum_{j=0}^{i}\sum_{t=0}^{k-i}(-1)^j\binom{i}{j}\binom{k-i}{t}m_{k-t-j,t+j}, i = 0, 1, 2, \ldots, k \tag{4}$$

in $m_{i,k-i} = E(R^iQ^{k-i}), i = 0, 1, 2, \ldots, k$, mixed moments of R and Q.
Letting γ,$\mathbf{m}$ respectively denote $(k + 1) \times 1$ vectors with $i-$th element $\gamma(i) = \gamma_{i0}\gamma_{0,k-i}, m(i) = m_{i,k-i}$ and $\mathbf{A} = (a_{i,k-i})$, the matrix of the numerical coefficients on the r.h.s of (4), we can rewrite $\gamma = \mathbf{Am}$.
The inverse of $\mathbf{A}$ can be accomplished by any statistical package for a given k and as a result, we obtain not only marginal moments of the max and the min but also their mixed moments of order k. For $k = 1, 2$, we have

$$m_{10} = (\gamma_{10} + \gamma_{01})/2, m_{01} = (\gamma_{10} - \gamma_{01})/2,$$

$$m_{11} = (\gamma_{02}\gamma_{20})/4, m_{20} = (\gamma_{02} + \gamma_{20} + 2\gamma_{01}\gamma_{10})/4, m_{02} = (\gamma_{02} + \gamma_{20} - 2\gamma_{01}\gamma_{10})/4.$$

Substituting $ER^i, EQ^i, i = 1, 2$ already discussed, yield first two moments of the max, the min and their covariance. It can be verified that they coincide with those given earlier with $\sigma_1^2 = \sigma_2^2$.
b) Here R and Q are correlated, however we have the same linear system except that the elements of γ are $\gamma(i) = \gamma_{i,k-i}, i = 0, 1, 2, \cdots, k$.
Now $\gamma_{0,k}, \gamma_{k,0}$ have already been discussed and $\gamma_{i,k-i}$ for i even, are known mixed moments of a bivariate normal r.v, needing no new computations. Thus it remains to find $\gamma(i)$, for i odd.
From the distribution of r.v. $(X_2 - X_1, X_2 + X_1)$, we can easily derive the joint p.d.f of Q, R. However to determine $\gamma_{i,k-i}$, it is convenient to find $E[Q^{k-i}|R = r]$ first. Accordingly we write the joint p.d.f. as

$$f(r, q) = [2\pi\tau_1\tau_2(1 - \rho^2)^{1/2}]^{-1}\{\phi[(r - \nu_1)/\tau_1)]\phi[\frac{q - \theta_1(r)}{\tau_2(1 - \rho^2)^{1/2}}] + \phi[(r + \nu_1)/\tau_1)]\phi[\frac{q - \theta_2(r)}{\tau_2(1 - \rho^2)^{1/2}}]\},$$

$\theta_1(r) = \nu_2 + \rho(\tau_2/\tau_1)(r - \nu_1)$ and $\theta_2(r) = \nu_2 - \rho(\tau_2/\tau_1)(r + \nu_1)$. From the p.d.f $f(r, q)$, it follows that

$$EQ^{k-i}|R = r) = E(\theta_j(r) + Z\tau_2(1 - \tau^2)^{1/2})^{k-i}, j = 1, 2 \tag{5}$$

where j refers to the j-th term of $f(r, q)$ and Z a standard normal r.v. Now evaluating (5) first which can be done with ease and then integrating the result obtained with the help of moments of R we get $\gamma_{i,k-i}$,e.g, $\gamma_{11} = E_1(R\theta_1(R)) + E_2(R\theta_2(R))$ where $E_i, i = 1, 2$ indicates the integral w.r.t i-th term of $f_R(r)$. Using first two moments of R we have

$$\gamma_{11} = (\nu_2 - \nu_1\rho\tau_2/\tau_1)ER + (\rho\tau_2/\tau_1)ER^2 + 2\rho\tau_2(\nu_1 - \tau_1)\phi(\nu_1/\tau_1) - 2(\rho\tau_2/\tau_1)(\nu_1^2 + \tau_1^2)\Phi(-\nu_1/\tau_1).$$

On solving linear system (4) with l.h.s modified appropriately in the correlated case, we can find moments of the max and the min and also their mixed moments without resorting either to their joint density or moment generating function.

REFERENCES

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RÉSUMÉ

Les moments du maximum (minimum) d'une variable aléatoire normale bidimensionelle sont obtenus par deux différentes méthodes. Le premier se fonde sur la relation entre la densité de probabilité du maximum et du minimum, tandis que le second demande d'exprimer cette statistique en termes de interval et de demiinterval.