An Empirical Bayes Estimation Problem

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1. Introduction

In empirical Bayes procedures the prior distribution of the unknown parameter, to be estimated, is assumed to be unknown. A procedure based on the kernel density estimate of the marginal density of the observable random variables is used by Mohammadzadeh (2000) to find an empirical Bayes estimator for the unknown parameter of one parameter exponential family. In this paper, another procedure based on spline density estimator (see Ciesielski, 1991) is used for the same distribution family. Then an empirical Bayes estimator for the scale parameter of Weibull distribution is derived. Let X be a random variable with conditional density

$$f(x|\theta) = m(x)\exp\{T(x)\theta - A(\theta)\} \tag{1}$$

where $\theta \in \Omega = \{\theta : \int \exp\{T(x)\theta - A(\theta)\}dx < \infty\}$, A is a real valued function of the parameter θ and T is a real valued statistic with non zero derivative with respect to x. If θ has a prior distribution G, then the marginal density of x is given by $f_G(x) = \int f(x|\theta)dG(\theta)$, and the Bayes estimator of θ, under the squared error loss, can be written as

$$\delta_G(x) = \frac{1}{T'(x)}\left[\frac{f'_G(x)}{f_G(x)} - \frac{m'(x)}{m(x)}\right]. \tag{2}$$

Since in empirical Bayes methods, G is assumed to be unknown, $\delta_G(x)$ cannot be obtained. Suppose however, that we have n previous independent observations $x_1, \ldots, x_n$ from distribution with densities $f(x_1|\theta_1), \ldots, f(x_n|\theta_n)$ where $\theta_1, \ldots, \theta_n$ are independent realizations of a random variable Θ with distribution G. These previous observations can be used to estimate $f_G(x)$ and $f'_G(x)$ and hence to obtain an estimate of $\delta_G(x)$. Spline density estimators with order $r \geq 2$, introduced by Ciesielski (1991), can be adopted to estimate $f_G(x)$ and $f'_G(x)$, respectively as

$$f_n(x) = \sum_{s=s_0}^{s_m} a_s F_s(x) \quad \text{and} \quad f'_n(x) = \sum_{s=s_0}^{s_m} a_s F'_s(x), \tag{3}$$

where $s_0 = \left[\frac{x_{min}}{h} - \nu\right] - r$, $s_m = \left[\frac{x_{max}}{h} - \nu\right] + 1$, the constant ν is zero or $\frac{1}{2}$ in terms of r is even or odd, and $[y]$ denotes the integer part of y,

$$a_s(x) = \frac{1}{nh}\sum_{j=1}^{n} F_s(x_j), \qquad s = s_0, s_0+1, \ldots, s_m,$$

and

$$F_s(x) = \sum_{k=1}^{r} N_{k,s}(x) I_{A_{k,s}}(x), \qquad F'_s(x) = \sum_{k=1}^{r} N'_{k,s}(x) I_{A_{k,s}}(x)$$

where, for $k = 1, \ldots, r$, $A_{k,s} = [(s+\nu+k-1)h\,, (s+\nu+k)h]$ and

$$\begin{aligned} N_{k,s}(x) &= \sum_{i=k}^{r} r \frac{(-1)^{r-i}}{i!(r-i)!}(s+\nu+i-\frac{x}{n})^{r-1}, \\ N'_{k,s}(x) &= \sum_{i=k}^{r} \frac{r(r-1)}{h} \frac{(-1)^{r-i+1}}{i!(r-i)!}(s+\nu+i-\frac{x}{n})^{r-2}. \end{aligned}$$

Krzykowski (1994) proposed $h = S\sqrt{\frac{6}{rn}}$, where S is the standard deviation of $x_1, \cdots, x_n$, as an optimal value for the window parameter h.

2. Empirical Bayes Estimation for Weibull Distribution

Suppose X has Weibull distribution with density

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}\exp\{-\theta x^{\alpha}\}, \qquad x > 0, \quad \alpha, \theta > 0.$$

For fixed value of the shape parameter α, this density function belongs to an exponential family of the form (**??**), with

$$m(x) = \alpha x^{\alpha-1}, \quad T(x) = -x^{\alpha}, \quad A(\theta) = -\log(\theta).$$

According to (**??**), the Bayes estimator of the scale parameter θ becomes

$$\delta_G(x) = \frac{-1}{\alpha x^{\alpha-1}} \left[\frac{f'_G(x)}{f_G(x)} + \frac{\alpha-1}{x} \right].$$

Based on the past observations of a given sample $X_1, \ldots, X_n$ and a current observation $X = x$ obtained from the distribution $f(x;\alpha,\theta)$, the functions $f_G(x)$ and $f'_G(x)$ can be estimated using the spline density estimators (**??**). Then an empirical Bayes estimator of θ is given by

$$\delta_n(x) = \frac{-1}{\alpha x^{\alpha-1}} \left[\frac{f'_n(x)}{f_n(x)} + \frac{\alpha-1}{x} \right].$$

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