On Fractional Factorial Designs With Two Levels

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1. Introduction

A matrix T of size $(m \times N)$ with two elements (say, 0 and 1) is called a balanced array (B-array) of strength t (t # m, T is called full strength if $t = m$) if in every t-rowed submatrix T_o of T, every $(t \times 1)$ vector $\underline{a}$ of weight $i (0 \leq i \leq t)$) appears with the same frequency $_i$ (say). The vector $\underline{m'} = (m_0, m_1, \ldots, m_t)$ is called the index set of the array, the number of rows m its constraints, and N its runs or treatment-combinations. Obviously B-array is reduced to an orthogonal array (O-array) if $m_i = m$ (for each i), and a B-array with $t = 2$ is an incidence matrix of an incomplete block design. Besides the importance of B-arrays in combinatorics, these arrays have been extensively used in the construction of symmetrical as well as asymmetrical factorial designs. A design T is said to be balanced if its variance-covariance matrix V_T is invariant under a permutation of its factor symbols, and is of resolution V if one can estimate all the effects up to and including two-factor interactions with the assumption that higher order interactions are negligible. B-arrays with $t = 4$, under certain conditions, give rise to balanced, resolution V designs.

In this paper we present some combinatorial results on B-arrays, and give some balanced resolution V designs with $m = 12$ and 96 # N # 99.

2. Some Combinatorial Results on *B*-arrays

Definition: A B-array T $(m \times N)$ is said to be trim if it does not contain any m-rowed column vector with weight 0 or m.

The following result is from Srivastava and Chopra (1971).

Theorem 2.1. Consider a resolution V balanced design T with index set $\underline{m'} = (m_0, m_1, m_2, m_3, m_4)$ and $m = 12$ then we have

$$tr\ V_t = \frac{c_o}{m_2} + \frac{c_1}{c_2} + \frac{c_3}{c_4},$$ where c_i ?s are polynomials in m_i ?s (clearly $m_2 > 0$).

$_2 \geq 2.$

Theorem 2.2. No trim B-array exists with $m = 12$, 96 # N # 99 and

Theorem 2.3. A trim B-array with $m_2 = 1$, $m = 12$, and $N (96 \leq N \leq 99)$ exists only if $N = 66$, 78, and 90.

Remark: An m-rowed B-array T_k obtained by writing all the distinct vectors of weight k $(0 \leq k \leq m)$ is clearly an array of full strength with index set $\left\{ \binom{m-t}{k-i}; i = 0,1,2,..,t \right\}$. Here $\binom{a}{b} = 0$ if $a < b$. For the array T_k, clearly $N = \binom{m}{k}$. The symbol $a_k T_k$ ($a_k > 0$, an integer) denotes the *B*-array obtained by writing each column of T_k exactly a_k times, and $\sum_{k=1}^{l} a_k T_k \; (l \leq m)$ stands for the juxtaposition of such arrays.

Theorem 2.4. The only trim *B*-arrays with $96 \leq N \leq 99$, $m = 12$, and $m_2 = 1$ are obtained from $a_1 T_1 + a_2 T_2 + a_{10} T_{10} + a_{11} T_{11}$ with $(a_2, a_{10}) = (1,0)$ or (0,1) and $(a_1, a_{11}) = (1,1)$, (2,0), or (0,2).

3. Construction of Trace-optimal Designs

It is quite clear that one has to construct appropriate *B*-arrays of strength four in order to construct balanced designs of resolution *V*. We rely on the results of the previous section to obtain these *B*-arrays. For a given *N* there are, in general, many *B*-arrays with $t = 4$. For each *N*, satisfying $96 \leq N \leq 99$, we obtain all the arrays, and compute the trace of the variance-covariance matrix for each, and present here the index set $\underline{m}'$ of the one with the minimum trace. We observe the index sets $\underline{m}'$ of the *B*-arrays giving rise to trace-optimal balanced designs are (34 + *k*, 8, 1, 2, 16) with $0 \leq k \leq 3$. For these designs the values of $tr\ V_T$ are 3.928, 3.912, 3.907, and 3.892 respectively.

For each design we list also the elements of the variance-covariance matrix (10,000 times their actual values which are: $var\ (\hat{m})$ = 1367, 1224, 1118, 1035; $cov\ (\hat{m}, \hat{A}_i)$ = 28, 24, 21, 18; $cov\ (\hat{m}, \hat{A}_{ij})$ = -35, -30, -27, -25; $var\ (\hat{A}_i)$ = 299, 298, 294, 293; $cov\ (\hat{A}_i, \hat{A}_j)$ = -24, -25, -25, -25; $cov\ (\hat{A}_i, \hat{A}_{ij})$ = 18, 18, 18, 18; $cov\ (\hat{A}_i, \hat{A}_{jk})$ = -4, -4, -4, -4; $var\ (\hat{A}_{ij})$ = 520, 519, 519, 519; $cov\ (\hat{A}_{ij}, \hat{A}_{ik})$ = -47, -47, -47, -47; $cov\ (\hat{A}_{ij}, \hat{A}_{kl})$ = 10.8, 10.6, 10.5, 10.5.

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RESUME

On a obtenu des plans fractionnaires equilibrés trace-optima de resolution *V* des séries de 2^{12} pour chaque valeur de *N* (le nombra de traitements) dans l ? intervalle $96 \leq N \leq 99$.