On Fractional Factorial Designs With Two Levels

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1. Introduction

A matrix $T$ of size $(m \times N)$ with two elements (say, 0 and 1) is called a balanced array ($B$-array) of strength $t$ ($t \neq m$, $T$ is called full strength if $t = m$) if in every $t$-rowed submatrix $T_0$ of $T$, every $(t \times 1)$ vector $\alpha$ of weight $i$ $(0 \leq i \leq t)$ appears with the same frequency $i$ (say). The vector $\mu' = (\mu_0, \mu_1, \ldots, \mu_t)$ is called the index set of the array, the number of rows $m$ its constraints, and $N$ its runs or treatment-combinations. Obviously $B$-array is reduced to an orthogonal array ($O$-array) if $\mu_i = \mu$ (for each $i$), and a $B$-array with $t = 2$ is an incidence matrix of an incomplete block design. Besides the importance of $B$-arrays in combinatorics, these arrays have been extensively used in the construction of symmetrical as well as asymmetrical factorial designs. A design $T$ is said to be balanced if its variance-covariance matrix $V_T$ is invariant under a permutation of its factor symbols, and is of resolution $V$ if one can estimate all the effects up to and including two-factor interactions with the assumption that higher order interactions are negligible. $B$-arrays with $t = 4$, under certain conditions, give rise to balanced, resolution $V$ designs.

In this paper we present some combinatorial results on $B$-arrays, and give some balanced resolution $V$ designs with $m = 12$ and $96 \# N \# 99$.

2. Some Combinatorial Results on $B$-arrays

Definition: A $B$-array $T (m \times N)$ is said to be trim if it does not contain any $m$-rowed column vector with weight 0 or $m$.

The following result is from Srivastava and Chopra (1971).

Theorem 2.1. Consider a resolution $V$ balanced design $T$ with index set $\mu' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ and $m = 12$ then we have

$$tr \ V_t = \frac{c_0}{\mu_2} + \frac{c_1}{\mu_2} + \frac{c_3}{\mu_2},$$

where $c_i \in \mathbb{R}$ are polynomials in $\mu_i \in \mathbb{R}$ (clearly $\mu_2 > 0$).

Theorem 2.2. No trim $B$-array exists with $m = 12, 96 \# N \# 99$ and

Theorem 2.3. A trim $B$-array with $\mu_2 = 1$, $m = 12$, and $N (96 \leq N \leq 99)$ exists only if $N = 66, 78, \text{ and } 90$.

Remark: An $m$-rowed $B$-array $T_k$ obtained by writing all the distinct vectors of weight $k$ $(0 \leq k \leq m)$ is
Clearly an array of full strength with index set \( \binom{m-t}{k-i} ; i = 0,1,2,\ldots,t \). Here \( \binom{a}{b} = 0 \) if \( a < b \). For the array \( T_k \), clearly \( N = \binom{m}{k} \). The symbol \( \alpha_k T_k \) \((\alpha_k > 0,\ \text{an integer})\) denotes the \( B \)-array obtained by writing each column of \( T_k \) exactly \( \alpha_k \) times, and \( \sum_{k=1}^{l} \alpha_k T_k \) \((l \leq m)\) stands for the juxtaposition of such arrays.

Theorem 2.4. The only trim \( B \)-arrays with \( 96 \leq N \leq 99 \), \( m = 12 \), and \( \mu_2 = 1 \) are obtained from \( \alpha_1 T_1 + \alpha_2 T_2 + \alpha_{10} T_{10} + \alpha_{11} T_{11} \) with \((\alpha_2,\alpha_{10}) = (1,0)\) or \((0,1)\) and \((\alpha_1,\alpha_{11}) = (1,1), (2,0), \) or \((0,2)\).

3. Construction of Trace-optimal Designs

It is quite clear that one has to construct appropriate \( B \)-arrays of strength four in order to construct balanced designs of resolution \( V \). We rely on the results of the previous section to obtain these \( B \)-arrays. For a given \( N \) there are, in general, many \( B \)-arrays with \( t = 4 \). For each \( N \), satisfying \( 96 \leq N \leq 99 \), we obtain all the arrays, and compute the trace of the variance-covariance matrix for each, and present here the index set \( \mu' \) of the one with the minimum trace. We observe the index sets \( \mu' \) of the \( B \)-arrays giving rise to trace-optimal balanced designs are \((34 + k, 8, 1, 2, 16)\) with \( 0 \leq k \leq 3 \). For these designs the values of \( \text{tr} \ V_T \) are 3.928, 3.912, 3.907, and 3.892 respectively.

For each design we list also the elements of the variance-covariance matrix (10,000 times their actual values which are: \( \text{var} \ (\hat{\mu}) = 1367, 1224, 1118, 1035; \text{cov} \ (\hat{\mu}, \hat{A}_j) = 28, 24, 21, 18; \text{cov} \ (\hat{A}_i, \hat{A}_j) = -35, -30, -27, -25; \text{var} \ (\hat{A}_j) = 299, 298, 294, 293; \text{cov} \ (\hat{A}_i, \hat{A}_j) = -24, -25, -25, -25; \text{cov} \ (\hat{A}_i, \hat{A}_j) = 18, 18, 18, 18; \text{cov} \ (\hat{A}_i, \hat{A}_j) = -4, -4, -4, -4; \text{var} \ (\hat{A}_j) = 520, 519, 519, 519; \text{cov} \ (\hat{A}_i, \hat{A}_j) = -7, -7, -7, -7; \text{cov} \ (\hat{A}_i, \hat{A}_j) = 10.8, 10.6, 10.5, 10.5.

REFERENCE


RESUME

On a obtenu des plans fractionnaires équilibrés trace-optima de resolution \( V \) des séries de \( 2^{12} \) pour chaque valeur de \( N \) (le nombre de traitements) dans l'intervalle \( 96 \leq N \leq 99 \).