Network of blocked tandem queues with withdrawal.

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ABSTRACT

We consider a network of two queues in tandem with one server in the first queue, and $n \geq 1$ in the second queue. Customers access the system through the first queue in accordance with Poisson input having parameter λ. Holding times are independent and identically distributed exponential random variables with rates μ_1 and μ_2 at the first and second queues respectively. A blocking phenomenon is observed in the system with empty buffer between stages, bringing about a withdrawal effect. We obtain the state probabilities and observe that the system has no product form.

Keywords: Blocking, tandem, queues, withdrawals.

1 Introduction

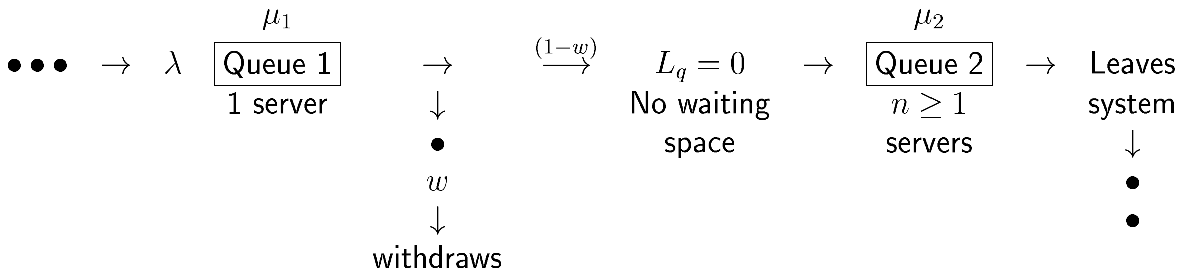
The earliest work on queues in tandem is due to Jackson (1954) with two service stations in tandem, and its extension to k (> 2) stations in Jackson (1956) with unlimited waiting spaces between service points. . However, when limitations of space are imposed between stages, then the problem of *blocking* is encountered, which imposes some difficulties in the analysis of such systems.

According to Foster and Perros (1980), blocking occurs when the flow of "customers" through one queue is momentarily stopped owing to a capacity limitation of the queue ahead. See also Akinsete (1999). A customer finishing service at the first queue is only allowed to proceed to the second queue if there is a free server in the second queue. Otherwise, the customer is blocked and stays in the first queue until there is an exit of at least a customer from the second queue.

There are numerous contributions in the literature on the analysis of tandem network of queues. See for example Makino (1964), Konheim and Reiser (1976), Langaris (1982), Leung (1993), Kumar (1996), Akinsete (2001), and Akinsete and Abe (1998). For such models, Wong *et. al.* (1977) had shown that the joint queue length process forms a two-dimensional finite, irreducible Markov process.

Here, we consider a network of two queues in tandem with one server in the first queue, and $n \geq 1$ servers in the second queue. Customers access the system through the first queue in accordance with Poisson input having parameter $\lambda > 0$. Service times are independent and identically distributed random variables with rates $\mu_1 > 0$ and $\mu_2 > 0$ at the first and second queues respectively, with a first-come, first- served discipline. There is no waiting space between the two service stages, i.e. the queue length (L_q) is zero in the second queue. Such a model is referred to as a 3-tuple(1,0,n)-model. See Langaris (1982).

The diagram below describes the model.



A two-stage tandem queue with blocking and withdrawal

2 Our main results

Let at time t, an incoming customer meet m and r customers in the first and second queues respectively, including those being served, where $m \geq 0$ and $0 \leq r \leq n$.
For state i, let

$$\xi_i(t) = \begin{cases} m, & i = 1 \\ r, & i = 2, \end{cases}$$

Define

$$P_{m,r}(t) = \text{Prob}\{\xi_1(t) = m,\ \xi_2(t) = r\}$$

and

$P'_{m,n}(t) = \text{Prob}\{\xi_1(t) = m,$ and the customer in service is blocked, $\xi_2(t) = n\}$

Let w denote the probability of withdrawal of a blocked customer who decides to renege from further service. Therefore, the random process

$$\xi = \{\xi_i(t) : i = 1, 2; 0 \leq t < \infty\}$$

assumes values in the strip $\{(m, r); 0 \leq m < \infty, 0 \leq r \leq n\}$ and constitutes a continuous time Markov chain.

Assuming that steady state exists, the global balance equations of the system become

$$(1) \quad (\lambda + \mu_1 + r\mu_2)P_{m,r} = \lambda P_{m-1,r} + \mu_1 P_{m+1,r-1} + (r+1)\mu_2 P_{m,r+1}; \quad 0 < m < \infty, 0 < r < n$$

with the following boundary equations:

$$(2) \quad (\lambda + \mu_1)P_{m,0} = \lambda P_{m-1,0} + \mu_2 P_{m,1}; \quad 0 < m < \infty$$

$$(3) \quad (\lambda + r\mu_2)P_{0,r} = \mu_1 P_{1,r-1} + (r+1)\mu_2 P_{0,r+1}; \quad 0 < r < n$$

$$(4) \quad (\lambda + \mu_1 + n\mu_2)P_{m,n} = \lambda P_{m-1,n} + \mu_1 P_{m+1,n-1} + \mu_1 w P_{m+1,n} + n\mu_2 P'_{m+1,n}; \quad 0 < m < \infty$$

$$(5) \quad (\lambda + n\mu_2)P_{0,n} = \mu_1 P_{1,n-1} + \mu_1 w P_{1,n} + n\mu_2 P'_{1,n}$$

$$(6) \quad (\lambda + n\mu_2)P'_{m,n} = \lambda P'_{m-1,n} + \mu_1(1-w)P_{m,n}$$

$$(7) \quad \lambda P_{0,0} = \mu_2 P_{0,1}$$

By Konheim and Reiser (1976, 1978), the system of equations (1) to (7) is homogeneous and hence always admits a bounded nonnull absolutely summable solution

$$\{P_{m,r}, P'_{m,n}\} = 0, \ \forall m, r, n,$$

which is also needed for the stability of the system. This requires that, $P_{m,r},\ P'_{m,n} \neq 0$ (for some $m, r,$ and n) and

$$\sum_{m,r,n} \mid P_{m,r} + P'_{m,n} \mid < \infty$$

Now define the following generating functions

$\mathcal{P}_r(z) = \sum_{0\leq m<\infty} P_{m,r} z^m, \quad 0 \leq r \leq n$ and $\mathcal{P}'_n(z) = \sum_{1\leq m<\infty} P'_{m,n} z^m$, where, $\mid z \mid\leq 1$.

These transform the system (1) to (7) to the following:

$$\begin{aligned} (8) \qquad \alpha_r \mathcal{P}_r(z) - \mu_1 z^{-1}\mathcal{P}_{r-1}(z) &- (r+1)\mu_2 \mathcal{P}_{r+1}(z) \\ &= \mu_1 P_{0,r} - \mu_1 z^{-1} P_{0,r-1};\ 0<r<n \end{aligned}$$

$$(9) \qquad \alpha_0 \mathcal{P}_0(z) - \mu_2 \mathcal{P}_1(z) = \mu_1 P_{0,0}$$

$$\begin{aligned} (10) \quad \alpha_n \mathcal{P}_n(z) - \mu_1 z^{-1}\mathcal{P}_{n-1}(z) &- n\mu_2 z^{-1}\mathcal{P}'_n(z) \\ &= \mu_1(1-wz^{-1})P_{0,n} - \mu_1 z^{-1} P_{0,n-1} \end{aligned}$$

$$(11) \qquad \alpha'_n \mathcal{P}'_n(z) - \mu_1(1-w)\mathcal{P}(z) = -\mu_1(1-w)P_{0,n}$$

where

$$\alpha_r = \lambda(1-z) + \mu_1(1-wz^{-1}\delta_r) + r\mu_2,$$

$$\delta_r = \begin{cases} 0, & 0 \leq r < n \\ 1, & r = n \end{cases}$$

and $\alpha'_n = \lambda(1-z) + n\mu_2$

The linear system of equations (8) to (11) can be expressed in the form

$$(12) \qquad A_n(z)\tilde{\mathcal{P}}(z) = B_n(z)\tilde{\mathcal{P}}_0$$

where $A_n(z)$ and $B_n(z)$ are tridiagonal $(n+2)\times(n+2)$ matrices with

$$\tilde{\mathcal{P}}(z) = (\mathcal{P}_0(z), \mathcal{P}_1(z), ..., \mathcal{P}_n(z), \mathcal{P}'_n(z))'$$

and

$$\tilde{\mathcal{P}}_0 = (P_{0,0}, P_{0,1}, ..., P_{0,n}, 0)'$$

being $(n+2)\times 1$ vectors.

Let (12) be written in the form

(13) $$A_n(z)\tilde{\mathcal{P}}(z) = \mathcal{D}(z)$$

where

$$\mathcal{D}(z) = (d_1(z), d_2(z), ..., d_{n+2}(z))'$$

with

$$d_j(z) = \mu_1(1 - wz^{-1}\delta_{j,0})P_{0,j-1} - \mu_1 z^{-1} P_{0,j-2};\ 1 \leq j \leq n+1$$

and

$$d_{n+2}(z) = -\mu_1(1-w)P_{0,n}.$$

$\delta_{j,0}$ is a Kronecker delta defined by

$$\delta_{j,0} = \begin{cases} 0, & 1 \leq j \leq n \\ 1, & j = n+1 \end{cases}$$

and $P_{0,k} = 0 \ \forall \ k < 0$

Following Burden and Faires (1989), (see also Westlake (1968)), the solution $\tilde{\mathcal{P}}(z) = \mathcal{X}$ of equation (13) is obtained by back substitution as follows:

(14) $$\left.\begin{array}{rcl} x_{n+2} & = & \tilde{d}_{n+2} = \tilde{\mathcal{P}}_{\mathbf{n}}(z) \\ x_i & = & \tilde{d}_i - \tilde{c}_i x_{i+1}, \quad i = n+1, n, ..., 1 \end{array}\right\}$$

where

(15) $$\tilde{c}_j = \frac{-jz\mu_2}{z\alpha_{j-1} + \mu_1\tilde{c}_{j-1}}, \quad j = 1, 2, ..., n;\ \tilde{c}_0 = 0$$

(16) $$\tilde{d}_j = \frac{z\mu_1 P_{0,j-1} - \mu_1 P_{0,j-2} + \mu_1\tilde{d}_{j-1}}{z\alpha_{j-1} - \frac{(j-1)\mu_1\mu_2 T_{j-3}}{T_{j-2}}\delta_{z,j}}, \quad j = 1, 2, ..., n$$

and

$$\alpha_j = \lambda(1-z) + \mu_1(1 - wz^{-1}\delta_j) + j\mu_2;$$

$$\delta_j = \begin{cases} 0, & j = 0, 1, ..., n+1 \\ 1, & j = n+2 \end{cases}$$

with

$$T_j = \alpha_j T_{j-1}\delta_{z,j} - j\mu_1\mu_2 T_{j-2},\ j \geq 1;\ \ T_0 = \alpha_0,\ T_{-1} = 1,\ T_j = 0 \ \text{ for } j \leq -2$$

We take,

$$\delta_{z,j} = \begin{cases} z, & \text{for odd } j \\ 1, & \text{for even } j \text{ and zero} \end{cases}$$

with $\tilde{d}_j = 0$ for $j \leq 0$ and $P_{0,j} = 0$ for $j < 0$.

We can now show that

$$\tilde{c}_{n+1} = \frac{-n\mu_2\beta(z)}{z(\alpha_n\beta(z) - n\mu_1\mu_2)}$$

where

$$\beta(z) = z\alpha_{n-1} + \mu_1\tilde{c}_{n-1},$$

$$\tilde{d}_{n+1} = \frac{H(.)}{z\lambda(z)(\alpha_n\beta(z) - n\mu_1\mu_2)}$$

and where

$$\begin{aligned} H(.) &= \mu_1\beta(z)\{z\lambda(z)(1 - wz^{-1})P_{0,n} + (z\mu_1 T_{n-2} - \lambda(z))P_{0,n-1} \\ &- \mu_1 T_{n-2}P_{0,n-2} + \mu_1 T_{n-2}\tilde{d}_{n-1}\} \end{aligned}$$

with

$$\lambda(z) = z\alpha_{n-1}T_{n-2} - (n-1)\mu_1\mu_2 T_{n-3}\delta_{z,n}.$$

Finally, we have

$$\tilde{d}_{n+2} = \frac{V(.)}{\lambda(z)[z\alpha'_n\varphi(z) - n\mu_1\mu_2(1-w)\beta(z)]}$$

where

$$\begin{aligned} V(.) &= (1-w)\{\mu_1 z\lambda(z)[\mu_1\beta(z)(1 - wz^{-1}) - \varphi(z)]P_{0,n} \\ &+ \mu_1^2\beta(z)(\mu_1 z T_{n-2} - \lambda(z))P_{0,n-1} - \mu_1^3\beta(z)T_{n-2}P_{0,n-2} + \mu_1^3\beta(z)T_{n-2}\tilde{d}_{n-1}\} \end{aligned}$$

and

$$\varphi(z) = \alpha_n\beta(z) - n\mu_1\mu_2$$

It is now possible to obtain

$$\mathcal{P}'_n(z) = \frac{(1-w)\sum_{j=0}^{n} D_j(z)P_{0,j}}{D_n(z)} \tag{17}$$

where,

$$D_0(z) = \mu_1^{n+1}(\mu_1 - T_0(z))$$

$$D_j(z) = \mu_1^{n-j+1}[z\mu_1(1 - wz^{-1})T_{j-1}(z) - T_j(z)], \quad j \neq 0,\ j \neq n$$

and

$$D_n(z) = \mid A_n(z) \mid = \alpha'_n T_n(z) - n\mu_1\mu_2(1-w)T_{n-1}(z)$$

And by means of successive back substitutions in equation (14), we can express $\mathcal{P}_r(z);\ 0 \leq r < n$, in terms of $\mathcal{P}'_n(z)$ in (17) and the boundary values $\{P_{0,r}\}$, from which the state probabilities can finally be retrieved.

3 A special case

We illustrate the results obtained above for $n = 1$. In this case we have a $(1,0,1)$-model with withdrawal. This presents us with a network of two queues in tandem with one server in each of the queues with unlimited waiting space in front of the first queue, and none between them. Using equations (15) and (16) we obtain the following:

$$(18) \quad \begin{cases} \tilde{c}_0 &= 0 = \tilde{c}_3 \\ \tilde{c}_1 &= -\mu_2\alpha_0^{-1} \\ \tilde{c}_2 &= -\mu_2\alpha_0 T_1^{-1} \\ \tilde{d}_0 &= 0 \\ \tilde{d}_1 &= \mu_1\alpha_0^{-1}P_{0,0} \\ \tilde{d}_2 &= \mu_1[(\mu_1-\alpha_0)P_{0,0} + z\alpha_0(1-wz^{-1})P_{0,1}]\}T_1^{-1} \\ \tilde{d}_3 &= \mu_1(1-w)[\mu_1(\mu_1-\alpha_0)P_{0,0} + (\mu_1 z\alpha_0(1-wz^{-1}) \\ &- T_1)P_{0,1}] \times (\Phi(z))^{-1} \end{cases}$$

where, $\Phi(z) = \alpha'_1 T_1 - \mu_1\mu_2\alpha_0(1-w)$.

By substituting corresponding expressions in (18) into (14), we obtain

$$\begin{aligned} \mathcal{P}'_1(z) &= \tilde{d}_3 \\ \mathcal{P}_1(z) &= \{\mu_1(\mu_1-\alpha_0)\alpha'_1 P_{0,0} + \mu_1\alpha_0[z\alpha'_1(1-wz^{-1}) \\ &- \mu_2(1-w)]P_{0,1}\}\Phi^{-1}(z) \\ \mathcal{P}_0(z) &= \{\mu_1[z\alpha_1\alpha'_1 - \mu_1\mu_2(1-w) - \alpha'_1\mu_2]P_{0,0} \\ &+ \mu_1\mu_2[z\alpha'_1(1-wz^{-1}) - \mu_2(1-w)]P_{0,1}\}\Phi^{-1}(z) \end{aligned}$$

And using the normalizing condition

$$\sum_{0 \leq r \leq n} \mathcal{P}_r(z=1) + \mathcal{P}'_n(z=1) = 1$$

along with (7), we can show that

$$(19) \qquad P_{0,0} = \frac{\mu_1\mu_2(\mu_1+\mu_2) - \lambda(\mu_1^2(1-w) + \mu_1\mu_2 + \mu_2^2)}{\mu_1\mu_2(\mu_1+\mu_2) + \lambda\mu_1(\mu_1 w + \mu_2)}$$

By definition,

$$P_{k,i} = \frac{1}{k!} d^{(k)} \mathcal{P}_i(z) \mid_{z=0}, \quad i = 0, 1$$

and

$$P'_{k,1} = \frac{1}{k!} d^{(k)} \mathcal{P}'_1(z) \mid_{z=0} .$$

We retrieve $\{P_{k,i}\}_k$ and $\{P'_{k,1}\}_k$ from their generating functions, noting that $P'_{0,i} = 0 = P'_{k,0}$ $\forall\, i$ and $k \geq 0$ for obvious reasons.

Finally, if we also set $m = 1$ in this model, we have a network with no waiting space in front of the first queue. The implication of this, is that arriving customers to the first queue are turned away. This occurs, either, when the system is blocked, or when a service is in progress at station one, even if the second queue is empty. In this case, the system (1) to (7) becomes

$$(20) \qquad \begin{cases} (\mu_1+\mu_2)P_{1,1} &= \lambda P_{0,1} \\ \mu_1 P_{1,0} &= \lambda P_{0,0} + \mu_2 P_{1,1} \\ (\lambda+\mu_2)P_{0,1} &= \mu_1 P_{1,0} + \mu_1 w P_{1,1} + \mu_2 P'_{1,1} \\ \mu_2 P'_{1,1} &= \mu_1(1-w)P_{1,1} \\ \lambda P_{0,0} &= \mu_2 P_{0,1} \end{cases}$$

The solution of (20) yields the following state probabilities:

$$(21) \qquad \begin{cases} P_{0,1} &= \frac{\lambda}{\mu_2} P_{0,0} \\ P_{1,0} &= \frac{\lambda(\lambda+\mu_1+\mu_2)}{\mu_1(\mu_1+\mu_2)} P_{0,0} \\ P_{1,1} &= \frac{\lambda^2}{\mu_2(\mu_1+\mu_2)} P_{0,0} \\ P'_{1,1} &= \frac{\mu_1(1-w)\lambda^2}{\mu_2^2(\mu_1+\mu_2)} P_{0,0} \\ P_{0,0} &= \frac{\mu_1\mu_2^2(\mu_1+\mu_2)}{B(\bullet)} \end{cases}$$

where,

$$B(\bullet) = \mu_1\mu_2\{(\mu_1+\mu_2)(\mu_2+\lambda) + \lambda^2\} + \lambda\{\mu_2^2(\lambda+\mu_1+\mu_2) + \lambda\mu_1^2(1-w)\}$$

4 Conclusion

We observe that by setting $w = 0$ in equation (21) we have the results obtained in Gross and Harris (1985), assuming homogeneous servers, i.e $\mu_1 = \mu_2$. We also observe that expressions (19) and (20) do not have a product form. According to Wong et al, the queues are not independent. The results show that our model generalises the result in the literature.

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