Convolution Semigroups on Hilbert Spaces

Byron Schmuland
Department of Mathematical Sciences
University of Alberta
Edmonton, Canada
schmu@stat.ualberta.ca

Wei Sun
Institute of Applied Mathematics
Chinese Academy of Sciences
100080 Beijing, China
wsun@stat.ualberta.ca

1. Introduction

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a separable Hilbert space E and $(\mu_t)_{t\geq 0}$ a family of probability measures on E. Using the notation $T_s\mu_t$ for the induced measure $\mu_t \circ T_s^{-1}$, we say that $(\mu_t)_{t\geq 0}$ is a (T_t)-*convolution semigroup* if

$$\mu_{t+s} = \mu_s * T_s\mu_t, \quad s, t \geq 0. \tag{1}$$

Such a family leads to a Markov semigroup $(p_t)_{t\geq 0}$ of operators, called a *(generalized) Mehler semigroup*, given by

$$p_t f(x) = \int_E f(T_t x + y)\, \mu_t(dy), \quad x \in E, f \in \mathcal{B}_b(E).$$

This generalization of the Ornstein-Uhlenbeck semigroup was studied in [1], [3]. There is also an important body of work ([2], [4]) that looks at equation (1) in the context of measure-valued branching processes.

In [5] it is proved that, if $(\mu_t)_{t\geq 0}$ solves (1), then the measures μ_t are infinitely divisible. By the Lévy-Khinchine theorem, each solution can be divided into three pieces: $\mu_t = \delta_{b_t} * g_t * j_t$, that is, a constant part, a Gaussian part, and a jump part. The Gaussian part g_t also solves (1), however b_t and j_t are more intimately tied together, and in general do not solve (1) separately.

2. Continuity

In [5] it is shown that $t \mapsto g_t * j_t$ is continuous, but that the constant part $t \mapsto b_t$ need not be. In a forthcoming paper, Schmuland and Sun will show that the map $t \mapsto g_t * j_t$ is, in fact, absolutely continuous and give an integral representation.

The following example shows that $t \mapsto b_t$ need not be absolutely continuous, even if it is continuous.

Example. Let $E = L^2([0,2\pi), dx/2\pi)$ and let T_t be the semigroup of shift operators (t mod 2π). For any $f \in E$ set $b_t = (I - T_t)f$ so that δ_{b_t} is a (T_t)-convolution semigroup. Take the inner product against f to obtain

$$\langle f, b_t \rangle = \|f\|^2 - \frac{1}{2\pi}\int_0^{2\pi} f(x-t)f(x)\,dx = \|f\|^2 - \left(|\hat{f}(0)|^2 + 2\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \cos(nt) \right).$$

Now let f be the function whose Fourier coefficients are given by

$$\hat{f}(n) = \begin{cases} \sqrt{2^{-k}} & \text{if } |n| = 2^k,\ k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\langle f, b_t \rangle = 2 - 2\sum_{k=1}^{\infty} 2^{-k} \cos(2^k t),$$

which is (up to a constant) Weierstrass's continuous but nowhere differentiable function.

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RESUME

Nous étudions une famille des mesures indéfiniment divisibles qui satisfont une certaine équation de convolution. Elles s'avèrent absolument continues, excepté la partie constante.