Optimal Two-sided Tests for Parameters of Cauchy Distribution.

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In this paper we deal with the Cauchy distribution with the density

$(f(x) =)\ f(x|\theta,\xi) = \xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1}$, for $-\infty<x<\infty$; $-\infty<\theta<\infty$ and $\xi>0$.

For inferences of the parameters of the Cauchy distribution we refer to G. Haas, L. Bain & C. Antle(1970). There, they used Monte Carlo method to obtain the distributions of the maximum likelihood estimates in order to get interval estimates and test the hypotheses. However, their methods are too complicated. In this paper the author uses the sample median for θ and median of log-transformed observation for ξ to test the hypotheses based on Lagrange's method. (The author has been working on inferences based on Lagrange's method since Y. Nogami(1992,1995;See also 2001.))

Based on a random sample $X_1, \dots ,X_n$ from f(x) we test the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ for a constant θ_0. Let $0<\alpha<1$. We call (U_1, U_2) a $(1-\alpha)$ interval estimate for the parameter γ if $P_\gamma[U_1<\gamma<U_2]=1-\alpha$.

For simplicity, we assume that n is odd and let n=2m+1 with m a nonnegative integer. Let $Y=X_{(m+1)}$, the sample median and an unbiased estimate for θ. We find the shortest $(1-\alpha)$ interval estimate for θ and, so, minimize $r_2-r_1(>0)$ subject to

(1) $P_\theta[r_1<Y-\theta<r_2]=1-\alpha$

or equivalently

(2) $P_\theta[F(r_1+\theta)<W<F(r_2+\theta)]=1-\alpha$

where $F(x)=\pi^{-1}\tan^{-1}\{\xi^{-1}(x-\theta)\}+2^{-1}$ for $-\infty<x<\infty$ and a known number ξ, and $W=F(Y)$. Since the p.d.f. of Y is given by

(3) $g_Y(y|\theta)=k(F(y))^m(1-F(y))^m f(y)\ (=h_w(F(y))f(y))$, for $-\infty<y<\infty$

where $k=\Gamma(2m+2)/(\Gamma(m+1))^2$, letting λ be a real number we define

$$L = r_2 - r_1 - \lambda\{\int_{F(r_1+\theta)}^{F(r_2+\theta)} h_w(w)dw - 1 + \alpha\}.$$

By Lagrange's method $\partial L/\partial r_1=0=\partial L/\partial r_2$, which leads to

(4) $h_w(F(r_1+\theta))f(r_1+\theta)=h_w(F(r_2+\theta))f(r_2+\theta)(=\lambda^{-1})$, $\forall\theta$.

Let $\beta(\alpha/2)$ be the positive number such that $\int_0^{\beta(\alpha/2)} h_w(w)dw=\alpha/2$ and assume that $0<\beta(\alpha/2)<2^{-1}$. Taking $F(r_1+\theta)=\beta(\alpha/2)$ and $F(r_2+\theta)=1-\beta(\alpha/2)$ we obtain that $\partial L/\partial\lambda=0$ (or (2) holds) and $r_1=-r_2(=-r)=-\xi\tan[(2^{-1}-\beta(\alpha/2))\pi]$. Thus, the shortest $(1-\alpha)$ interval estimate for θ is given by $(Y-r,Y+r)$. Inverting this interval for θ_0 we obtain the acceptance region $(\theta_0-r,\ \theta_0+r)$ of our test of size α.

Letting $\zeta(\theta) = \int_{\theta_0-r}^{\theta_0+r} g_Y(y|\theta)dy$ we want to show that $\zeta(\theta)$ is maximized at $\theta=\theta_0$ and $\zeta(\theta_0) = 1-\alpha$. (1) and (4) applied to our test leads to $\zeta(\theta_0)=1-\alpha$ and $[d\zeta(\theta)/d\theta]_{\theta=\theta_0}= [g_Y(\theta_0-r|\theta) - g_Y(\theta_0+r|\theta)]_{\theta=\theta_0}=0$. Thus, it is sufficient to show

Theorem: $[d^2\zeta(\theta)/d\theta^2]_{\theta=\theta_0}<0$. (A proof is presented at the seminor.)

We now test the hypotheses $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ with some constant ξ_0. We assume that θ is known. For simplicity, let n=2m+1. Let $Z=\log_e|X-\theta|$. Define $Y=Z_{(m+1)}$ which is an unbiased estimate for $\log_e \xi$. In the similar method to the above, we obtain the (1-α) interval estimate (r_1e^Y, r_2e^Y) for ξ where $r_1=[\tan\{2^{-1}\ \pi(1-\beta(\alpha/2))\}]^{-1}$ and $r_2=[\tan\{2^{-1}\ \pi\beta(\alpha/2)\}]^{-1}$. Inverting this interval for ξ_0 we obtain the acceptance region $(\log_e\xi_0 - \log_e r_2 , \log_e\xi_0 - \log_e r_1)$ of our test of size α. $\zeta(\xi)$ of this test is also maximized at $\xi=\xi_0$ and $\zeta(\xi_0)=1-\alpha$.

(When n=2m, we omit the analyses.)

REFERENCE

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RESUME

G. Haas, L. Bain et C. Antle(1970,Biometrika) discutent certaines inférences de la loi de Cauchy C(θ, ξ), dont la densité de probabilité est donnée par $f(x|\theta,\xi) = \xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1}$ $(-\infty<x<\infty; -\infty<\theta<\infty; \xi>0)$. *Ils ont obtenu, par la méthode de Monte Carlo, la distribution des estimateurs à vraisemblance maximale pour acquérir des estimations d'intervalles et éprouver les hypothèses. Leurs procédés, cependant, sont trop compliqués.*

Dans cet article l'auteur emploie, pour θ, la médiane d'un échantillon et, pour ξ, la médiane des logarithmes d'observations pour obtenir des estimations d'intervalles par la méthode de Lagrange et invertit ces dernières pour éprouver l'hypothèse $H_0: \theta = \theta_0$ *(ou* $H^*_0 : \xi=\xi_0$ *) vis-à-vis de l'hypothèse alternative* $H_1: \theta \neq \theta_0$ *(ou* $H^*_0 : \xi\neq\xi_0$ *) avec une constante* θ_0 *(ou* ξ_0 *). Dans les deux cas, la fonction de puissance se minimise à* H_0 *(ou* H^*_0 *).*