Two-Sample Rank Tests for Skewness and Kurtosis

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1. Introduction

Consider two independent random samples $X_1, ..., X_m$ and $Y_1, ..., Y_n$ drawn from unknown continuous distributions $F$ and $G$. To test the null hypothesis of the equality of the two distributions $H_0: F(x) = G(x) \forall x$ against the alternative $H_1: \exists \theta, F(x) = G((x - \theta)) \forall x$, where the location shift $\theta \neq 0$, one may use the Mann and Whitney’s (1947) procedure, whose test statistic is the sum of the the ranks of the $Y_i$ in the combined sample. This statistic is distribution-free under $H_0$.

If we are interested to test a change of scale between $F$ and $G$ when the two distributions have possibly a different location, that is if we wish to test the null hypothesis $H_1$ against the alternative

$$H_2: \exists \theta, \sigma > 0, F(x) = G((x - \theta)/\sigma) \forall x$$

where the scale ratio $\sigma \neq 1$, we may use the procedure proposed by Moses (1963). It consists to rearrange the $X_i$ and $Y_i$ in random order, to define for $i = 1, 3, 5, ..$

$$U_i = |X_{i+1} - X_i|,$$

$$V_i = |Y_{i+1} - Y_i|,$$

and to apply the Mann and Whitney’s test on the $U_i$ and $V_i$. If a sample size is odd, the last observation is discarded. Again, we have here an exact test.

In this note, we are interested to define a strictly nonparametric procedure for testing the null hypothesis $H_2$ that $F$ and $G$ have the same shape, against the alternative that the two distributions differ in skewness or in kurtosis.

2. Skewness and Kurtosis Orderings

Let $X$ and $Y$ be two continuous random variables with distribution $F$ and $G$ respectively. The one-sided version of Mann and Whitney’s (1947) test is known to be powerful against the alternative hypothesis that $X$ is stochastically larger than $Y$, noted here $Y \prec X$. On the other hand, Oja (1981) defines $X$ to be stochastically more skew to the right than $Y$ if

$$\frac{Y(3) - Y(2)}{Y(3) - Y(1)} \geq \frac{X(3) - X(2)}{X(3) - X(1)}$$

where $X(1), X(2), X(3)$ and $Y(1), Y(2), Y(3)$ are ordered samples of size three from $F$ and $G$. Similarly, $X$ is stochastically more flattened (or has stochastically less kurtosis) than $Y$ if

$$\frac{Y(3) - Y(2)}{Y(4) - Y(1)} \geq \frac{X(3) - X(2)}{X(4) - X(1)}$$
where $X_{(1)}, ..., X_{(4)}$ and $Y_{(1)}, ..., Y_{(4)}$ are ordered samples of size four from $F$ and $G$. This suggests the following nonparametric procedure for testing whether two distributions differ in skewness or in kurtosis, similar to the rank test for dispersion proposed by Moses (1963). Note that Behboodian (1989) proposed a procedure to test the symmetry of a distribution which also considers independent triples of observations, and which requires to divide by three the sample size.

Let rearrange the $X_i$ and $Y_i$ in random order, and define for $i = 1, 4, 7, ...$

$$U_i = \frac{X_{(i+2)} - X_{(i+1)}}{X_{(i+2)} - X_{(i)}}$$

$$V_i = \frac{Y_{(i+2)} - Y_{(i+1)}}{Y_{(i+2)} - Y_{(i)}}$$

The proposed procedure consists to apply the Mann and Whitney’s test on the $U_i$ and $V_i$. If $m$ or $n$ are not multiples of three, some observations are discarded. The test statistic obtained is distribution-free under the null hypothesis $H_2$ defined in Section 1. A rejection of $H_2$ using this procedure means that the two distributions differ in skewness.

Similarly one could apply the Mann and Whitney’s test on the

$$U_i = \frac{X_{(i+2)} - X_{(i+1)}}{X_{(i+3)} - X_{(i)}}$$

$$V_i = \frac{Y_{(i+2)} - Y_{(i+1)}}{Y_{(i+3)} - Y_{(i)}}$$

where $i = 1, 5, 9, ....$ Again, some observations are discarded if $m$ or $n$ are not multiples of four. Rejecting the null hypothesis $H_2$ by applying this procedure means that the two distributions differ in kurtosis.

We will apply this methodology to find the direction of relationship between two variables in a simple regression problem as in Dodge and Rousson (2001).

REFERENCES


RESUME

Dans cet article nous construisons les tests de rang pour comparer le skewness et le kurtosis de deux distributions. Nos procédures sont strictement non-paramétriques.

*Mots clés:* Kurtosis, Nonparametric tests, Rank tests, Skewness.